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A Procedure for Numerical Evaluation of the Performance of a TM_{01} Circular to TE_{10} Rectangular Waveguide Mode Converter

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Two identical rectangular waveguides are fed by means of two symmetrically placed apertures in a circular waveguide. Power is transmitted from the TM_{01} mode in the circular waveguide to the TE_{10} dominant mode in each rectangular waveguide. A previously described method of evaluating this transmitted power is refined to obtain an algorithm for which a computer program was written. Several plots of transmitted power and magnitude of electric field in one of the apertures are presented. The computer program that was used to obtain the numerical data for these plots will be described and listed in a subsequent report.

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Chapter 1

Introduction

There is, as shown in Fig. 1.1, a circular waveguide which is closed at one end. Two symmetrically placed apertures in the lateral wall of this waveguide are backed by rectangular waveguides of identical dimensions. The dimensions of the waveguides are such that only the TE_{11} and TM_{01} modes can propagate in the circular waveguide and that only the TE_{10} mode can propagate in the rectangular waveguides. The problem is, as stated in [1, Chapter 1], to find out how much of the power of an incident TM_{01} wave in the circular waveguide is reflected in the circular waveguide and how much of this power is transmitted into the rectangular waveguides.

In this report, the analytical results of [1] for this problem are manipulated into expressions suitable for evaluation by means of a digital computer. These analytical results are not derived here; they are merely referred to. For this reason, the reader of this report should obtain a copy of reference [1].

A computer program was written in FORTRAN. Some numerical results obtained by using this computer program are presented. The computer program will be described and listed in a forthcoming report.

As is shown in Fig. 1.2, the interiors of the left-hand rectangular waveguide, the right-hand rectangular waveguide, and the circular waveguide are called regions 1, 2, and 3, respectively. The electromagnetic field in region 1 is radiated by $\underline{M}^{(1)}$. The field in region 2 is radiated by $\underline{M}^{(2)}$. The field in region 3 is radiated by the combination of $\underline{J}^{\text{imp}}$, $-\underline{M}^{(1)}$, and $-\underline{M}^{(2)}$. The magnetic currents in Fig. 1.2 are supposed to be loacated right on (infinitesimal distances from either side) of the closing conductors. The finite displacement of these magnetic currents from the closing conductors in Fig. 1.2 is only for

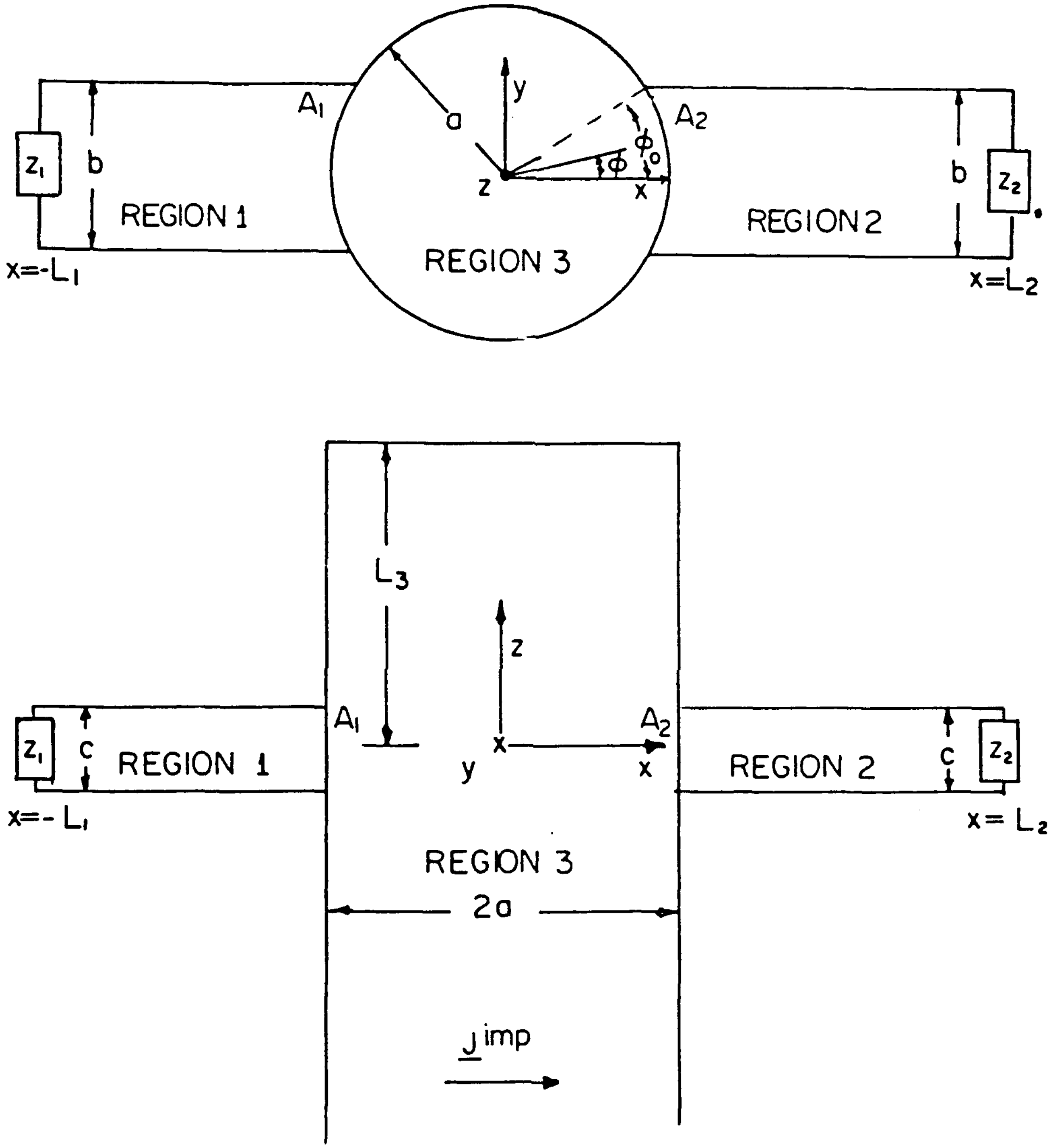


Fig. 1.1. Top and side views of the TM_{01} to TE_{10} mode converter.

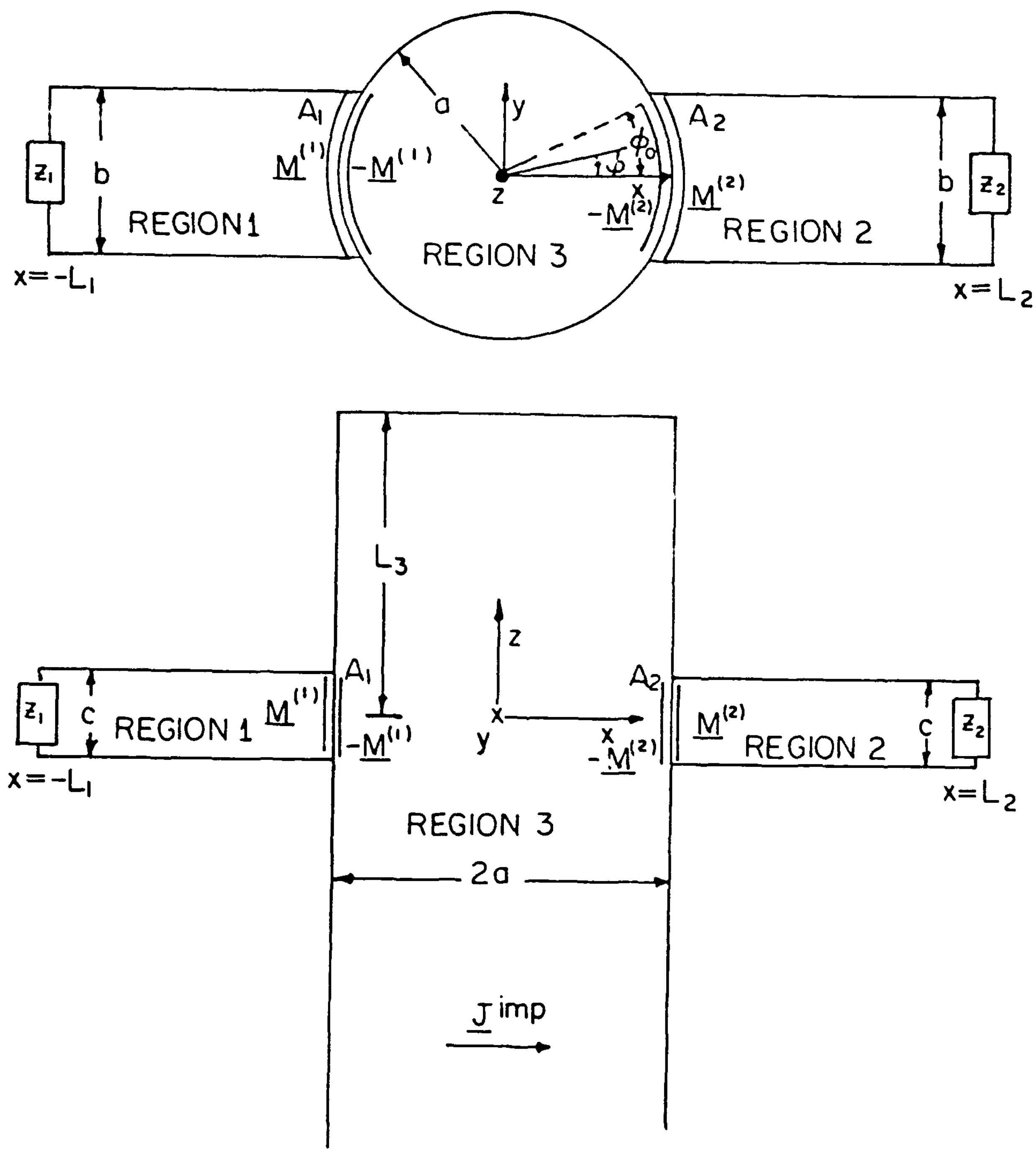


Fig. 1.2. Top and side views of the situation equivalent to that of Fig. 1.1.

the purpose of illustration. The magnetic currents $\underline{M}^{(1)}$ and $\underline{M}^{(2)}$ are given by [1, eqs. (2.11) and (2.12)] in which the V's are the elements of the column vector on the left-hand side of [1, eq. (2.22)]. In [1, eq. (2.22)], Y^1, Y^2, and Y^3 are the admittance matrices for regions 1, 2, and 3, respectively. The column vector on the right-hand side of [1, eq. (2.22)] is called the excitation vector. The matrices Y^1 and Y^2 are treated in Chapter 2, the matrix Y^3 is treated in Chapter 3, and the excitation vector is treated in Chapter 4.

After solving [1, eq. (2.22)] for the V's which determine $\underline{M}^{(1)}$ and $\underline{M}^{(2)}$ according to [1, eqs. (2.11) and (2.12)], we find the electromagnetic fields in regions 1, 2, and 3. Expressions for the fields in regions 1 and 2 are obtained in Chapter 5. The field in region 3 due to the combination of $-\underline{M}^{(1)}$ and $-\underline{M}^{(2)}$ is considered in Chapter 6. An expression is obtained for this field below the apertures where $z < -c/2$. Expressions are also obtained for this field in the apertures. In Chapter 7, numerical results are presented for the power transmitted into the rectangular waveguides, the power reflected back into the circular waveguide, and the magnitudes of the ϕ- and z-components of the electric field in one of the apertures when a TE_{01} wave is incident in the circular waveguide and when the loads Z_1 and Z_2 that terminate the rectangular waveguides are both matched loads.

In Appendix A, the expansion functions $\{\underline{M}_{mn}^{1TM}, \underline{M}_{mn}^{1TE}, \underline{M}_{mn}^{2TM}, \underline{M}_{mn}^{2TE}\}$ are ordered so that each one of them can be identified by means of a single positive integer rather than the attached combination of subscripts and superscripts. In Appendix B, a numerical procedure for obtaining roots of Bessel functions and their derivatives is described. Heretofore, the excitation has simply been a z-traveling TM_{01} wave in the circular waveguide. In Appendix C, the response due to this excitation is used to find the response due to excitation by a transverse sheet of TM_{01} electric current between two impedance loads as shown in Fig. C.1.

Chapter 2

The Admittance Matrices for the Rectangular Waveguides

The admittance matrice for region 1, the left-hand rectangular waveguide in Fig. 1.2, is Y^1 given by [1, eq. (2.25)] where the Y's on the right-hand side of [1, eq. (2.25)] are approximated by $\hat{Y}$'s given by [1, eqs. (3.44)–(3.47)]. The superscripts on the $\hat{Y}$'s are the same as those on the Y's.

For convenience, [1, eqs. (3.45) and (3.46)] are repeated:

$$\hat{Y}_{ij}^{1,1TE,1TM} = 0 \tag{2.1}$$

$$\hat{Y}_{ij}^{1,1TM,1TE} = 0. \tag{2.2}$$

In [1, eqs. (3.44) and (3.47)], we have [1, eq. (A.12)]

$$\gamma_{pq} = \begin{cases} j\beta_{pq}, & k_{pq} < k \\ \sqrt{k_{pq}^2 - k^2}, & k_{pq} \geq k \end{cases} \tag{2.3}$$

where $k = \omega\sqrt{\mu\epsilon}$ in which ω is the angular frequency. Moreover, μ and ϵ are, respectively, the permeability and permittivity of the medium in regions 1, 2, and 3. In (2.3),

$$k_{pq} = \sqrt{\left(\frac{p\pi}{b}\right)^2 + \left(\frac{q\pi}{c}\right)^2} \tag{2.4}$$

$$\beta_{pq} = \sqrt{k^2 - k_{pq}^2}. \tag{2.5}$$

There is a correspondence between each pair of integers (p,q) used in [1, eqs. (3.44) and (3.47)] and the subscript j in [1, eqs. (3.44) and (3.47)]. This correspondence is described in Appendix A. If $(p,q) = (1,0)$, then, because the TE_{10} mode propagates in the rectangular waveguide, $k > k_{10}$ and substitution of (2.3) into [1, eq. (3.47)] and subsequent multiplication by $-j\eta$ where $\eta = \sqrt{\mu/\epsilon}$ gives

$$-j\eta \hat{Y}_{ij}^{1,1TE,1TE} = -j\frac{\beta_{10}(\cos\beta_{10}x_1 + jZ_1Y_{10}^{TE}\sin\beta_{10}x_1)}{k(j\sin\beta_{10}x_1 + Z_1Y_{10}^{TE}\cos\beta_{10}x_1)}\delta_{ij}, \quad (p,q) = (1,0) \tag{2.6}$$

where

$$x_1 = L_1 - x_o \tag{2.7}$$

$$x_o = \frac{a\sin\phi_o}{\phi_o} \tag{2.8}$$

$$\phi_o = \sin^{-1}\frac{b}{2a}. \tag{2.9}$$

Y_{10}^{TE} is the characteristic admittance of the TE_{10} mode in the rectangular waveguide [1, eq. (A.25)]

$$Y_{10}^{TE} = \frac{\beta_{10}}{k\eta}, \tag{2.10}$$

and δ_{ij} is the Kronecker delta function given by

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}. \tag{2.11}$$

The identities [2, formulas 654.6 and 654.7] were used to obtain (2.6). In (2.6), the subscript j is not to be confused with the other j's. Each of these other j's is $\sqrt{-1}$. Because the TE_{pq} mode does not propagate in the rectangular waveguide when $(p,q) \neq (1,0)$, γ_{pq} is real when $(p,q) \neq (1,0)$ and, from [1, eq. (3.47)],

$$-j\eta\hat{Y}_{ij}^{1,1TE,1TE} = -\frac{\gamma_{pq}}{k}\delta_{ij}, \quad (p,q) \neq (1,0). \tag{2.12}$$

The factor $-j$ on the left-hand side of (2.12) has rendered the right-hand side of (2.12) real. The factor η on the left-hand sides of (2.6) and (2.12) has rendered the right-hand sides of (2.6) and (2.12) independent of η. Multiplication of [1, eq. (3.44)] by $-j\eta$ gives

$$-j\eta\hat{Y}_{ij}^{1,1TM,1TM} = \frac{k}{\gamma_{pq}}\delta_{ij}. \tag{2.13}$$

Because all the TM modes in the ctangular waveguide are evanescent, γ_{pq} is real in (2.13) so that the right-hand side of (2.13) is real.

The admittance matrix for region 2, the right-hand rectangular waveguide in Fig. 1.2, is Y^2 given by [1, eq. (2.27)] where the Y's on the right-hand side are approximated by $\hat{Y}$'s given by [1, eqs. (3.49)–(3.52)]. Similar to (2.1), (2.2), (2.6), (2.7), (2.12), and (2.13), we have

$$\begin{aligned}
\hat{Y}_{ij}^{2,2TE,2TM} &= 0 && (2.14)\\
\hat{Y}_{ij}^{2,2TM,2TE} &= 0 && (2.15)\\
-j\eta\hat{Y}_{ij}^{2,2TE,2TE} &= -j\frac{\beta_{10}(\cos\beta_{10}x_2 + jZ_2Y_{10}^{TE}\sin\beta_{10}x_2)}{k(j\sin\beta_{10}x_2 + Z_2Y_{10}^{TE}\cos\beta_{10}x_2)}\delta_{ij}, \\
& \qquad (p,q) = (1,0) && (2.16)\\
x_2 &= L_2 - x_o && (2.17)\\
-j\eta\hat{Y}_{ij}^{2,2TE,2TE} &= -\frac{\gamma_{pq}}{k}\delta_{ij},\ \ (p,q) \neq (1,0) && (2.18)\\
-j\eta\hat{Y}_{ij}^{2,2TM,2TM} &= \frac{k}{\gamma_{pq}}\delta_{ij}. && (2.19)
\end{aligned}$$

Chapter 3

The Admittance Matrix for the Circular Waveguide

The admittance matrix for region 3, the circular waveguide in Fig. 1.2, is Y^3 given by [1, eq. (2.29)] where the Y's on the right-hand side of [1, eq. (2.29)] are given by [1, eqs. (4.93), (4.111), (4.112), and (4.113)] in which T, S_1, S_2, S_3, S_4, and S_5 are given by [1, eqs. (4.94)–(4.99)]. The previously mentioned equations for the Y's are recast as

$$-j\eta Y_{ij}^{3,\alpha TM,\gamma TM} = \sum_{r=0}^{\infty}\sum_{s=1}^{\infty} \hat{T}\{W_8\hat{S}_1 + W_9\hat{S}_3 - W_{10}\hat{S}_4 - W_{11}\hat{S}_5\} \quad (3.1)$$

$$-j\eta Y_{ij}^{3,\alpha TE,\gamma TM} = \sum_{r=0}^{\infty}\sum_{s=1}^{\infty} \hat{T}\{W_9\hat{S}_1 - W_8\hat{S}_3 - W_{11}\hat{S}_4 + W_{10}\hat{S}_5\} \quad (3.2)$$

$$-j\eta Y_{ij}^{3,\alpha TM,\gamma TE} = \sum_{r=0}^{\infty}\sum_{s=1}^{\infty} \hat{T}\{W_{10}\hat{S}_1 + W_{11}\hat{S}_3 + W_8\hat{S}_4 + W_9\hat{S}_5\} \quad (3.3)$$

$$-j\eta Y_{ij}^{3,\alpha TE,\gamma TE} = \sum_{r=0}^{\infty}\sum_{s=1}^{\infty} \hat{T}\{W_{11}\hat{S}_1 - W_{10}\hat{S}_3 + W_9\hat{S}_4 - W_8\hat{S}_5\} \quad (3.4)$$

where

$$\hat{T} = \frac{2\pi}{ka(k_{mn}b)(k_{pq}b)}\sqrt{\frac{\epsilon_m\epsilon_n\epsilon_p\epsilon_q}{16}} \quad (3.5)$$

$$W_8 = \frac{nqb}{c} \quad (3.6)$$

$$W_9 = mq \quad (3.7)$$

$$W_{10} = np \tag{3.8}$$

$$W_{11} = \frac{mpc}{b} \tag{3.9}$$

$$\hat{S}_1 = (z_1 - z_2)\hat{\phi}^{\alpha\gamma 2} \tag{3.10}$$

$$\hat{S}_3 = z_3\hat{\phi}^{\alpha\gamma 1} \tag{3.11}$$

$$\hat{S}_4 = z_4\hat{\phi}^{\alpha\gamma 3} \tag{3.12}$$

$$\hat{S}_5 = z_5\hat{\phi}^{\alpha\gamma 4} \tag{3.13}$$

$$\hat{\phi}^{\alpha\gamma 1} = (-1)^{\alpha}\phi^{\alpha\gamma 1} \tag{3.14}$$

$$\hat{\phi}^{\alpha\gamma 2} = \phi^{\alpha\gamma 2} \tag{3.15}$$

$$\hat{\phi}^{\alpha\gamma 3} = (-1)^{\gamma}\phi^{\alpha\gamma 3} \tag{3.16}$$

$$\hat{\phi}^{\alpha\gamma 4} = (-1)^{\alpha+\gamma}\phi^{\alpha\gamma 4} \tag{3.17}$$

$$z_1 = \frac{(ka)^2(k_{rs}^{TM}a)^2 J_r'^2(k_{rs}^{TM}a)\hat{z}^{(1)}}{x_{rs}^2 J_{r+1}^2(x_{rs})}\left(\frac{\epsilon_r}{2}\right) \tag{3.18}$$

$$z_2 = \frac{r^2 J_r^2(k_{rs}^{TE}a)\hat{z}^{(2)}}{(x_{rs}'^2 - r^2)J_r^2(x_{rs}')} \tag{3.19}$$

$$z_3 = \frac{(k_{rs}^{TE}a)^2 r J_r^2(k_{rs}^{TE}a)\hat{z}^{(3)}}{(x_{rs}'^2 - r^2)J_r^2(x_{rs}')}\left(\frac{\sin\phi_o}{\phi_o}\right) \tag{3.20}$$

$$z_4 = \frac{(k_{rs}^{TE}a)^2 r J_r^2(k_{rs}^{TE}a)\hat{z}^{(4)}}{(x_{rs}'^2 - r^2)J_r^2(x_{rs}')}\left(\frac{\sin\phi_o}{\phi_o}\right)\left(\frac{\epsilon_r}{2}\right) \tag{3.21}$$

$$z_5 = \frac{(k_{rs}^{TE}a)^4 J_r^2(k_{rs}^{TE}a)\hat{z}^{(5)}}{(x_{rs}'^2 - r^2)J_r^2(x_{rs}')}\left(\frac{\sin\phi_o}{\phi_o}\right)^2\left(\frac{\epsilon_r}{2}\right) \tag{3.22}$$

$$\hat{z}^{(1)} = \frac{4z^{(1)}}{c^2\gamma_{rs}^{TM}a} \tag{3.23}$$

$$\hat{z}^{(2)} = \frac{4z^{(2)}\gamma_{rs}^{TE}a}{c^2} \tag{3.24}$$

$$\hat{z}^{(3)} = \frac{4z^{(3)}}{c^2} \tag{3.25}$$

$$\hat{z}^{(4)} = \frac{4z^{(4)}}{c^2} \tag{3.26}$$

$$\hat{z}^{(5)} = \frac{4z^{(5)}}{c^2\gamma_{rs}^{TE}a}. \tag{3.27}$$

In (3.5)–(3.9), j determines p and q in the manner described in Appendix A; i determines m and n in the same manner. In obtaining (3.1)–(3.4), we started the index r of summation in [1, eqs. (4.96) and (4.97)] at 0 instead of 1. This was possible because the $r = 0$ terms so introduced are zero. We multiplied [1, eq. (4.93)] by $\sqrt{\epsilon_m \epsilon_n \epsilon_p \epsilon_q / 16}$, [1, eq. (4.111)] by $\sqrt{\epsilon_p \epsilon_q / 4}$, and [1, eq. (4.112)] by $\sqrt{\epsilon_m \epsilon_n / 4}$. This was possible because none of the indices m, n, p, and q which appear in these multipliers is ever zero.

3.1 Evaluation of the $\hat{\phi}$'s in (3.10)–(3.13)

The $\hat{\phi}$'s in (3.10)–(3.13) are given by (3.14)–(3.17). Substitution of [1, eqs. (E.31)–(E.34)] into (3.14)–(3.17) gives

$$\hat{\phi}^{\alpha\gamma 1} = (-1)^{\alpha+\gamma} \left\{ \phi_p^{(2)} \phi^{\alpha 2 \gamma 1} - \phi_p^{(1)} \phi^{\alpha 2 \gamma 2} \right\} \tag{3.28}$$

$$\hat{\phi}^{\alpha\gamma 2} = \phi_p^{(2)} \phi^{\alpha 1 \gamma 2} + \phi_p^{(1)} \phi^{\alpha 1 \gamma 1} \tag{3.29}$$

$$\hat{\phi}^{\alpha\gamma 3} = \phi_p^{(4)} \phi^{\alpha 1 \gamma 1} - \phi_p^{(3)} \phi^{\alpha 1 \gamma 2} \tag{3.30}$$

$$\hat{\phi}^{\alpha\gamma 4} = (-1)^{\alpha+\gamma} \left\{ \phi_p^{(4)} \phi^{\alpha 2 \gamma 2} + \phi_p^{(3)} \phi^{\alpha 2 \gamma 1} \right\}. \tag{3.31}$$

For $\gamma = \alpha$, we have [1, eqs. (E.46)–(E.49)]

$$\phi^{\alpha 1 \gamma 1} = \phi_m^{(1)}, \quad \gamma = \alpha \tag{3.32}$$

$$\phi^{\alpha 2 \gamma 1} = \phi_m^{(3)}, \quad \gamma = \alpha \tag{3.33}$$

$$\phi^{\alpha 1 \gamma 2} = \phi_m^{(2)}, \quad \gamma = \alpha \tag{3.34}$$

$$\phi^{\alpha 2 \gamma 2} = \phi_m^{(4)}, \quad \gamma = \alpha. \tag{3.35}$$

For $\gamma \neq \alpha$, we have [1, eqs. (E.53)–(E.56)]

$$\phi^{\alpha 1 \gamma 1} = (-1)^r \left\{ \phi_m^{(2)} \sin \frac{rb}{x_o} - \phi_m^{(1)} \cos \frac{rb}{x_o} \right\}, \quad \gamma \neq \alpha \tag{3.36}$$

$$\phi^{\alpha 2 \gamma 1} = (-1)^r \left\{ \phi_m^{(4)} \sin \frac{rb}{x_o} - \phi_m^{(3)} \cos \frac{rb}{x_o} \right\}, \quad \gamma \neq \alpha \tag{3.37}$$

$$\phi^{\alpha 1 \gamma 2} = (-1)^r \left\{ \phi_m^{(2)} \cos \frac{rb}{x_o} + \phi_m^{(1)} \sin \frac{rb}{x_o} \right\}, \quad \gamma \neq \alpha \tag{3.38}$$

$$\phi^{\alpha 2 \gamma 2} = (-1)^r \left\{ \phi_m^{(4)} \cos \frac{rb}{x_o} + \phi_m^{(3)} \sin \frac{rb}{x_o} \right\}, \quad \gamma \neq \alpha. \tag{3.39}$$

In (3.28)–(3.31), the ϕ_p's are given by [1, eqs. (E.23)–(E.26)]

$$\phi_p^{(1)} = \frac{b}{2x_o}\left\{\frac{\sin A^-}{A^-} - \frac{\sin A^+}{A^+}\right\} \tag{3.40}$$

$$\phi_p^{(2)} = \frac{b}{2x_o}\left\{\frac{\sin^2(A^-/2)}{(A^-/2)} + \frac{\sin^2(A^+/2)}{(A^+/2)}\right\} \tag{3.41}$$

$$\phi_p^{(3)} = \frac{b}{2x_o}\left\{-\frac{\sin^2(A^-/2)}{(A^-/2)} + \frac{\sin^2(A^+/2)}{(A^+/2)}\right\} \tag{3.42}$$

$$\phi_p^{(4)} = \frac{b}{2x_o}\left\{\frac{\sin A^-}{A^-} + \frac{\sin A^+}{A^+}\right\} \tag{3.43}$$

where

$$A^+ = p\pi + \frac{rb}{x_0} \tag{3.44}$$

$$A^- = p\pi - \frac{rb}{x_0}. \tag{3.45}$$

The ϕ's in (3.36)–(3.39) are given by the right-hand sides of (3.40)–(3.43) with A^+ and A^- replaced by the right-hand sides of (3.44) and (3.45) with p replaced by m.

3.2 Evaluation of the z's in (3.10)–(3.13)

The z's in (3.10)–(3.13) are given by (3.18)–(3.22). In (3.18)–(3.22), ϵ_r is Neumann's number given by [1, eq. (B.9)]

$$\epsilon_r = \begin{cases} 1, & r = 0 \\ 2, & r = 1, 2, \cdots \end{cases}, \tag{3.46}$$

J_r is the Bessel function of the first kind of order r, x_{rs} is the s^{th} root of J_r, J_r' is the derivative of J_r with respect to its argument, and x_{rs}' is the s^{th} root of J_r'. The roots $\{x_{rs},\ r = s = 1, 2, \cdots\}$ and $\{x_{rs}',\ r = 1, 2, \cdots\}$ are ordered such that

$$0 < x_{r1} < x_{r2} < x_{r3} \cdots \tag{3.47}$$

$$0 < x_{r1}' < x_{r2}' < x_{r3}' \cdots \tag{3.48}$$

Still in (3.18)-(3.22), we have [1, eqs. (B.7) and (B.41)]

$$k_{rs}^{TM} = \frac{x_{rs}}{a} \tag{3.49}$$

$$k_{rs}^{TE} = \frac{x'_{rs}}{a}. \tag{3.50}$$

Since $J_r(x_{rs}) = 0$, we have [3, formula 9.1.27]

$$J'_r(x_{rs}) = -J_{r+1}(x_{rs}). \tag{3.51}$$

Substitution of (3.49)-(3.51) into (3.18)-(3.22) yields

$$z_1 = (ka)^2 \left(\frac{\epsilon_r}{2}\right) \hat{z}^{(1)} \tag{3.52}$$

$$z_2 = \frac{r^2 \hat{z}^{(2)}}{x'^2_{rs} - r^2} \tag{3.53}$$

$$z_3 = \frac{r x'^2_{rs} \hat{z}^{(3)}}{x'^2_{rs} - r^2} \left(\frac{\sin \phi_o}{\phi_o}\right) \tag{3.54}$$

$$z_4 = \frac{r x'^2_{rs} \hat{z}^{(4)}}{x'^2_{rs} - r^2} \left(\frac{\epsilon_r}{2}\right) \left(\frac{\sin \phi_o}{\phi_o}\right) \tag{3.55}$$

$$z_5 = \frac{x'^4_{rs} \hat{z}^{(5)}}{x'^2_{rs} - r^2} \left(\frac{\epsilon_r}{2}\right) \left(\frac{\sin \phi_o}{\phi_o}\right)^2. \tag{3.56}$$

The $\hat{z}$'s in (3.52)-(3.56) are given by (3.23)-(3.27) where [1, eqs. (B.24) and (B.53)]

$$\gamma_{rs}^{TM} a = \begin{cases} j\beta_{rs}^{TM} a & , \ x_{rs} < ka \\ \sqrt{x_{rs}^2 - (ka)^2} & , \ x_{rs} \geq ka \end{cases} \tag{3.57}$$

$$\gamma_{rs}^{TE} a = \begin{cases} j\beta_{rs}^{TE} a & , \ x'_{rs} < ka \\ \sqrt{x'^2_{rs} - (ka)^2} & , \ x'_{rs} \geq ka \end{cases} \tag{3.58}$$

where

$$\beta_{rs}^{TM} a = \sqrt{(ka)^2 - x_{rs}^2} \tag{3.59}$$

$$\beta_{rs}^{TE} a = \sqrt{(ka)^2 - x'^2_{rs}}. \tag{3.60}$$

From [1, eqs. (F.60), (F.61), (F.76), (F.81), (F.118), and (F.119)],

$$z^{(1)} = \frac{j}{4}\left(D^{TM}G^{TM} + c^2F^{TM}\right) \tag{3.61}$$

$$z^{(2)} = \frac{j}{4}\left(D^{TE}G^{TE} + c^2F^{TE}\right) \tag{3.62}$$

$$z^{(3)} = \frac{1}{4}\left(D^{(3)}G^{(3)} + c^2F^{(3)}\right) \tag{3.63}$$

$$z^{(4)} = -\frac{1}{4}\left(D^{TE}G^{(4)} + c^2F^{(4)}\right) \tag{3.64}$$

$$z^{(5)} = \frac{\gamma_{rs}^{TE} z_{ss}^{TE}}{(k_{rs}^{TE})^2} + \frac{j}{4}\left(D^{(3)}G^{(4)} + c^2F^{(5)}\right) \tag{3.65}$$

where the D's, the G's, the F's, and z_{ss}^{TE} are dealt with in [1, Appendix F]. In view of [1, eq. (F.83)] and (3.50), substitution of (3.61)–(3.65) into (3.23)–(3.27) gives

$$\hat{z}^{(1)} = \frac{j\left(F^{TM} + \hat{G}_q^{TM}\hat{D}_n^{TM}\right)}{\gamma_{rs}^{TM}a} \tag{3.66}$$

$$\hat{z}^{(2)} = j\left(F^{TE} + \hat{G}_q^{TE}\hat{D}_n^{TE}\right)\gamma_{rs}^{TE}a \tag{3.67}$$

$$\hat{z}^{(3)} = F^{(3)} + \hat{G}_q^{TE}\hat{D}_n^{(3)} \tag{3.68}$$

$$\hat{z}^{(4)} = -\left(F^{(4)} + \hat{G}_q^{(4)}\hat{D}_n^{TE}\right) \tag{3.69}$$

$$\hat{z}^{(5)} = \frac{4az_{ss}^{TE}}{c^2x_{rs}'^2} + \frac{j\left(F^{(5)} + \hat{G}_q^{(4)}\hat{D}_n^{(3)}\right)}{\gamma_{rs}^{TE}a} \tag{3.70}$$

where

$$\hat{G}_q^{\delta} = \frac{G^{\delta}}{c}, \quad \delta = TM, TE \tag{3.71}$$

$$\hat{D}_n^{\delta} = \frac{D^{\delta}}{c}, \quad \delta = TM, TE \tag{3.72}$$

$$\hat{D}_n^{(3)} = \frac{D^{(3)}}{c} \tag{3.73}$$

$$\hat{G}_q^{(4)} = \frac{G^{(4)}}{c}. \tag{3.74}$$

3.2.1 The TM Quantity $\hat{z}^{(1)}$ for $x_{rs} < ka$

With $x_{rs} < ka$, substitution of (3.57) into (3.66) gives

$$\hat{z}^{(1)} = \frac{F^{TM} + \hat{G}_q^{TM} \hat{D}_n^{TM}}{\beta_{rs}^{TM} a}. \tag{3.75}$$

Substituting [1, eq. (F.87)] into (3.71), we obtain

$$\begin{aligned} \hat{G}_q^\delta = & \frac{\sin(q^{\delta-}c)\cos(\beta_{rs}^\delta L_3^+) - 2\sin^2\left(\frac{q^{\delta-}c}{2}\right)\sin(\beta_{rs}^\delta L_3^+)}{q^{\delta-}c} \\ & + \frac{\sin(q^{\delta+}c)\cos(\beta_{rs}^\delta L_3^+) + 2\sin^2\left(\frac{q^{\delta+}c}{2}\right)\sin(\beta_{rs}^\delta L_3^+)}{q^{\delta+}c} \end{aligned} \tag{3.76}$$

where δ may be either TM or TE and where [1, eqs. (F.11), (F.23), and (F.24)]

$$L_3^+ = L_3 + \frac{c}{2} \tag{3.77}$$

$$q^{\delta-}c = q\pi - \beta_{rs}^\delta c \tag{3.78}$$

$$q^{\delta+}c = q\pi + \beta_{rs}^\delta c. \tag{3.79}$$

Using (3.77)-(3.79) and [2, formulas 403.02, 401.03, and 401.04], we reduce (3.76) to

$$\hat{G}_q^\delta = \left\{ \frac{\sin\left(\frac{q^{\delta+}c}{2}\right)}{\frac{q^{\delta+}c}{2}} + (-1)^q \frac{\sin\left(\frac{q^{\delta-}c}{2}\right)}{\frac{q^{\delta-}c}{2}} \right\} \cos\left(\beta_{rs}^\delta L_3 - \frac{q\pi}{2}\right) \tag{3.80}$$

Substituting [1, eq. (F.85)] into (3.72), we obtain

$$\hat{D}_n^\delta = \left\{ \frac{-j\sin(n^{\delta-}c) - 2\sin^2\left(\frac{n^{\delta-}c}{2}\right)}{n^{\delta-}c} + \frac{-j\sin(n^{\delta+}c) + 2\sin^2\left(\frac{n^{\delta+}c}{2}\right)}{n^{\delta+}c} \right\} e^{-j\beta_{rs}^\delta L_3^+} \tag{3.81}$$

where $n^{\delta-}c$ and $n^{\delta+}c$ are given by (3.78) and (3.79) with q replaced by n. In the same manner as we reduced (3.76) to (3.80), we can reduce (3.81) to

$$
\begin{aligned}
\hat{D}_n^\delta \;=\; & -\left\{ \frac{\sin\left(\dfrac{n^{\delta+}c}{2}\right)}{\dfrac{n^{\delta+}c}{2}} + (-1)^n \frac{\sin\left(\dfrac{n^{\delta-}c}{2}\right)}{\dfrac{n^{\delta-}c}{2}} \right\} \\
& \cdot \left\{ \sin\left(\beta_{rs}^\delta L_3 - \frac{n\pi}{2}\right) + j\cos\left(\beta_{rs}^\delta L_3 - \frac{n\pi}{2}\right) \right\} \qquad (3.82)
\end{aligned}
$$

As for the quantity F^{TM} in (3.75), we have [1, eq. (F.79)]

$$
\begin{aligned}
F^\delta \;=\; & -f(n^{\delta-}c, -q^{\delta-}c) + f(n^{\delta+}c, q^{\delta-}c) \\
& -f(n^{\delta-}c, q^{\delta+}c) + f(n^{\delta+}c, -q^{\delta+}c) \qquad (3.83)
\end{aligned}
$$

where δ may be either TM or TE and [1, eq. (F.97)]

$$
f(x,y) = \begin{cases}
-\dfrac{\sin x}{yx} & , \begin{cases} x+y \neq 0 \\ |y| > \frac{\pi}{2} \end{cases} \\[2ex]
\dfrac{(-1)^I \sin y}{yx} & , \begin{cases} x+y \neq 0 \\ |y| \leq \frac{\pi}{2} \end{cases} \\[2ex]
\dfrac{y - \sin y}{y^2} & , \begin{cases} x+y = 0 \\ |y| > 0.1 \end{cases} \\[2ex]
\dfrac{y}{3!} - \dfrac{y^3}{5!} + \dfrac{y^5}{7!} & , \begin{cases} x+y = 0 \\ |y| \leq 0.1 \end{cases}
\end{cases} \qquad (3.84)
$$

where I is the integer that satisfies

$$x + y = I\pi . \qquad (3.85)$$

When $x_{rs} < ka$, the TM quantity $\hat{z}^{(1)}$ is now given by (3.75) in which $\beta_{rs}^{TM}a$, $\hat{G}_q^{TM}$, $\hat{D}_n^{TM}$, and F^{TM} are given by (3.59), (3.80), (3.82) and (3.83), respectively.

3.2.2 The TM Quantity $\hat{z}^{(1)}$ for $x_{rs} > ka$

When $x_{rs} > ka$, we proceed to evaluate expression (3.66) for $\hat{z}^{(1)}$, which contains the quantities $\hat{G}_q^{TM}, \hat{D}_n^{TM}$, and F^{TM}. These quantities with the superscript TM replaced by δ are, according to (3.71), (3.72), and [1, eqs. (F.104), (F.100), (F.101), and (F.107)–(F.109)], given by

$$\hat{G}_q^\delta = c_q\left\{\sinh(gL_3^+) - (-1)^q \sinh\left(g(L_3^+ - c)\right)\right\} \tag{3.86}$$

$$\hat{D}_n^\delta = 2jc_n e^{-g(L_3^+ - \frac{c}{2})} \left\{ \begin{array}{ll} -\sinh\left(\frac{gc}{2}\right) & ,\ n \text{ even} \\ \cosh\left(\frac{gc}{2}\right) & ,\ n \text{ odd} \end{array} \right\} \tag{3.87}$$

$$F^\delta = j \left\{ \begin{array}{ll} (-1)^n c_n c_q \sinh(gc) & ,\ q \neq n \\ (-1)^n c_n c_q \sinh(gc) - c_q & ,\ q = n \neq 0 \\ \dfrac{4\{\sinh(gc) - gc\}}{(gc)^2} & ,\ q = n = 0 \end{array} \right\} \tag{3.88}$$

where

$$g = \gamma_{rs}^\delta \tag{3.89}$$

$$c_n = \frac{2\gamma_{rs}^\delta c}{(n\pi)^2 + (\gamma_{rs}^\delta c)^2}. \tag{3.90}$$

Furthermore, c_q is the right-hand side of (3.90) with n replaced by q. The truncated series approximation inherent in [1, eq. (F.110)] is introduced later in this section. The case where $x_{rs} = ka$ is not allowed because, if $x_{rs}a = ka$, then, according to (3.57), $\gamma_{rs}^{TM}a$ would be zero so that division by $\gamma_{rs}^{TM}a$ in (3.66) would be impossible.

Substituting (3.77) into (3.86) and using [2, formulas 651.06 and 651.07], we obtain

$$\hat{G}_q^\delta = 2c_q \left\{ \begin{array}{ll} \sinh(gc/2)\cosh(gL_3) & ,\ q \text{ even} \\ \sinh(gL_3)\cosh(gc/2) & ,\ q \text{ odd} \end{array} \right\}. \tag{3.91}$$

Substitution of (3.77) into (3.87) gives

$$\hat{D}_n^\delta = 2jc_n e^{-gL_3} \left\{ \begin{array}{ll} -\sinh(gc/2) & ,\ n \text{ even} \\ \cosh(gc/2) & ,\ n \text{ odd} \end{array} \right\}. \tag{3.92}$$

Combining (3.88), (3.91), and (3.92) and using [2, formulas 652.12 and 654.5] to simplify the result, we obtain

$$j\left(F^\delta + \hat{G}_q^\delta \hat{D}_n^\delta\right) = c_{nq} + c_n \left\{ \begin{array}{ll} 1 & ,\ q = n \neq 0 \\ 0 & ,\ \text{otherwise} \end{array} \right\} \tag{3.93}$$

where

$$c_{nq} = \left\{ \begin{array}{lll} c_n c_q z_{ee} & , & \left\{ \begin{array}{l} n \text{ even} \\ q \text{ even} \\ q \neq n \end{array} \right. \\ c_n c_q z_{ee} & , & \left\{ \begin{array}{l} n \text{ even} \\ q = n \neq 0 \end{array} \right. \\ c_n c_q z_o & , & q = n = 0 \\ c_n c_q z_{oe} & , & \left\{ \begin{array}{l} n \text{ odd} \\ q \text{ even} \end{array} \right. \\ c_n c_q z_{oe} & , & \left\{ \begin{array}{l} n \text{ even} \\ q \text{ odd} \end{array} \right. \\ c_n c_q z_{oo} & , & \left\{ \begin{array}{l} n \text{ odd} \\ q \text{ odd} \end{array} \right. \end{array} \right\} . \tag{3.94}$$

In (3.94),

$$z_{ee} = 2\left\{e^{-2gL_3}\sinh\frac{gc}{2} - e^{-\frac{gc}{2}}\right\}\sinh\frac{gc}{2} \tag{3.95}$$

$$z_o = z_{ee} + gc \tag{3.96}$$

$$z_{oe} = -e^{-2gL_3}\sinh gc \tag{3.97}$$

$$z_{oo} = 2\left\{e^{-2gL_3}\cosh\frac{gc}{2} - e^{-\frac{gc}{2}}\right\}\cosh\frac{gc}{2} . \tag{3.98}$$

The value of $\sinh(gc/2)$ is excessively large when $gc/2$ is only moderately large. Moreover, z_o approaches zero more rapidly than gc as gc approaches zero. To avoid computational difficulties, we replace (3.95)–(3.98) by

$$z_{ee} = \left\{ \begin{array}{ll} e^{-gc} - e^{-2gL_3} - 1 + \frac{1}{2}\left\{e^{-g(2L_3-c)} + e^{-g(2L_3+c)}\right\} & , \ gc \geq 1 \\ 2\left\{e^{-2gL_3}\sinh\frac{gc}{2} - e^{-\frac{gc}{2}}\right\}\sinh\frac{gc}{2} & , \ gc < 1 \end{array} \right. \tag{3.99}$$

$$z_o = \left\{ \begin{array}{ll} e^{-gc} - e^{-2gL_3} - 1 + \frac{1}{2}\left\{e^{-g(2L_3-c)} + e^{-g(2L_3+c)}\right\} + gc, & gc \geq 1 \\ 2\left\{e^{-2gL_3}\sinh\frac{gc}{2} - e^{-\frac{gc}{2}}\right\}\sinh\frac{gc}{2} + gc & , \ 0.01 \leq gc < 1 \\ 2\Big\{ \left(2e^{-\frac{gc}{4}}\sinh\frac{gc}{4} + e^{-2gL_3}\sinh\frac{gc}{2}\right)\sinh\frac{gc}{2} & \\ -\left(\frac{\left(\frac{gc}{2}\right)^3}{3!} + \frac{\left(\frac{gc}{2}\right)^5}{5!} + \frac{\left(\frac{gc}{2}\right)^7}{7!}\right)\Big\} & , \ gc < 0.01 \end{array} \right. \tag{3.100}$$

$$z_{oe} = \begin{cases} \frac{1}{2}\left\{e^{-g(2L_3+c)} - e^{-g(2L_3-c)}\right\} & , \; gc \geq 1 \\ -e^{-2gL_3}\sinh gc & , \; gc < 1 \end{cases} \tag{3.101}$$

$$z_{oo} = \left\{ \begin{array}{ll} e^{-2gL_3} - e^{-gc} - 1 + \frac{1}{2}\left\{e^{-g(2L_3-c)} + e^{-g(2L_3+c)}\right\}, & gc \geq 1 \\ 2\left\{e^{-2gL_3}\cosh\frac{gc}{2} - e^{-\frac{gc}{2}}\right\}\cosh\frac{gc}{2} & , \; gc < 1 \end{array} \right\}. \tag{3.102}$$

We used the approximation [2, formula 657.1]

$$\sinh x - x = \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \tag{3.103}$$

to obtain z_o of (3.100) for $gc < 0.01$. From Fig. 1.2, $L_3 \geq c/2$ so that all the arguments of the exponentials in (3.99)-(3.102) are less than or equal to zero. The exponentials in (3.99)-(3.102) will be excessively small when gL_3 is only moderately large. However, this will not cause any difficulty if we use a computing system which treats an underflow by setting the number equal to zero and proceeding without an error message. The right-hand side of (3.94) cannot be evaluated when $gc = 0$ and when $n = 0$ or $q = 0$ because $c_n = 2/(gc)$ when $n = 0$ and $c_q = 2/(gc)$ when $q = 0$. However, the right-hand side of (3.94) remains finite as gc approaches zero when $n = 0$ or $q = 0$.

When $x_{rs} > ka$, the TM quantity $\hat{z}^{(1)}$ is now given by (3.66) in which $\gamma_{rs}^{TM}a$ and $j(F^{TM} + \hat{G}_q^{TM}\hat{D}_n^{TM})$ are given by (3.57) and (3.93), respectively. In (3.93), c_n and c_{nq} are given by (3.90) and (3.94), respectively. In (3.94), z_{ee}, z_o, z_{oe}, and z_{oo} are given by (3.99)-(3.102) where g is given by (3.89).

3.2.3 The TE Quantities $\hat{z}^{(2)}$–$\hat{z}^{(5)}$ for $x'_{rs} < ka$

When $x'_{rs} < ka$, $\gamma_{rs}^{TE}a$ is given by (3.58) so that expressions (3.67)-(3.70) for $\hat{z}^{(2)}$–$\hat{z}^{(5)}$ become

$$\hat{z}^{(2)} = -\left(F^{TE} + \hat{G}_q^{TE}\hat{D}_n^{TE}\right)\beta_{rs}^{TE}a \tag{3.104}$$

$$\hat{z}^{(3)} = F^{(3)} + \hat{G}_q^{TE}\hat{D}_n^{(3)} \tag{3.105}$$

$$\hat{z}^{(4)} = -\left(F^{(4)} + \hat{G}_q^{(4)}\hat{D}_n^{TE}\right) \tag{3.106}$$

$$\hat{z}^{(5)} = \frac{F^{(5)} + \hat{G}_q^{(4)}\hat{D}_n^{(3)}}{\beta_{rs}^{TE}a} + \left\{ \begin{array}{ll} \frac{2a}{cx_{rs}'^2} & , \; n = q \neq 0 \\ 0 & , \; \text{otherwise} \end{array} \right\}. \tag{3.107}$$

We obtained (3.107) by substituting [1, eq. (F.121)] for z_{ss}^{TE}.

In (3.104), $\beta_{rs}^{TE}a$, $\hat{G}_q^{TE}$, $\hat{D}_n^{TE}$, and F^{TE} are given by (3.60), (3.80), (3.82), and (3.83), respectively. As for $\hat{G}_q^{(4)}$ in (3.106) and (3.107), substitution of [1, eq. (F.122)] into (3.74) gives

$$\hat{G}_q^{(4)} = \frac{\sin(q^{TE-}c)\cos(\beta_{rs}^{TE}L_3^+) - 2\sin^2\left(\frac{q^{TE-}c}{2}\right)\sin(\beta_{rs}^{TE}L_3^+)}{q^{TE-}c}$$
$$- \frac{\sin(q^{TE+}c)\cos(\beta_{rs}^{TE}L_3^+) + 2\sin^2\left(\frac{q^{TE+}c}{2}\right)\sin(\beta_{rs}^{TE}L_3^+)}{q^{TE+}c}. \quad (3.108)$$

Note that the right-hand side of (3.108) is that of (3.76) with δ replaced by TE and with the sign of the coefficient of $1/(q^{\delta+}c)$ changed. If we recall also that (3.76) reduced to (3.80), we see that (3.108) reduces to

$$\hat{G}_q^{(4)} = \left\{ -\frac{\sin\left(\frac{q^{TE+}c}{2}\right)}{\left(\frac{q^{TE+}c}{2}\right)} + (-1)^q \frac{\sin\left(\frac{q^{TE-}c}{2}\right)}{\left(\frac{q^{TE-}c}{2}\right)} \right\} \cos(\beta_{rs}^{TE}L_3 - \frac{q\pi}{2}). \quad (3.109)$$

As for $\hat{D}_n^{(3)}$ in (3.105) and (3.107), substitution of [1, eq. (F.86)] into (3.73) gives

$$\hat{D}_n^{(3)} = \left\{ \frac{j\sin(n^{TE-}c) + 2\sin^2\left(\frac{n^{TE-}c}{2}\right)}{n^{TE-}c} + \frac{-j\sin(n^{TE+}c) + 2\sin^2\left(\frac{n^{TE+}c}{2}\right)}{n^{TE+}c} \right\}$$
$$\cdot e^{-j\beta_{rs}^{TE}L_3^+}. \quad (3.110)$$

The right-hand side of (3.110) is that of (3.81) with δ replaced by TE and with the sign of the coefficient of $1/(n^{\delta-}c)$ changed; recalling that (3.81) reduced to (3.82), we see that (3.110) reduces to

$$\hat{D}_n^{(3)} = \left\{ -\frac{\sin\left(\frac{n^{TE+}c}{2}\right)}{\left(\frac{n^{TE+}c}{2}\right)} + (-1)^n \frac{\sin\left(\frac{n^{TE-}c}{2}\right)}{\left(\frac{n^{TE-}c}{2}\right)} \right\}$$

$$\cdot \left\{ \sin\left(\beta_{rs}^{TE} L_3 - \frac{n\pi}{2}\right) + j\cos\left(\beta_{rs}^{TE} L_3 - \frac{n\pi}{2}\right)\right\}. \quad (3.111)$$

The F's in (3.105)–(3.107) are given by [1, eqs. (F.84), (F.123), and (F.124)]

$$\begin{aligned} F^{(3)} &= f(n^{TE-}c, -q^{TE-}c) + f(n^{TE+}c, q^{TE-}c) \\ &\quad + f(n^{TE-}c, q^{TE+}c) + f(n^{TE+}c, -q^{TE+}c) \end{aligned} \quad (3.112)$$

$$\begin{aligned} F^{(4)} &= -f(n^{TE-}c, -q^{TE-}c) + f(n^{TE+}c, q^{TE-}c) \\ &\quad + f(n^{TE-}c, q^{TE+}c) - f(n^{TE+}c, -q^{TE+}c) \end{aligned} \quad (3.113)$$

$$\begin{aligned} F^{(5)} &= f(n^{TE-}c, -q^{TE-}c) + f(n^{TE+}c, q^{TE-}c) \\ &\quad - f(n^{TE-}c, q^{TE+}c) - f(n^{TE+}c, -q^{TE+}c) \end{aligned} \quad (3.114)$$

where f is given by (3.84).

3.2.4 The TE Quantities $\hat{z}^{(2)}$–$\hat{z}^{(5)}$ for $x'_{rs} > ka$

When $x'_{rs} > ka$, we proceed to evaluate expressions (3.67)–(3.70) for $\hat{z}^{(2)} - -\hat{z}^{(5)}$. Expression (3.67) is

$$\hat{z}^{(2)} = j(F^{TE} + \hat{G}_q^{TE} \hat{D}_n^{TE})\gamma_{rs}^{TE} a \quad (3.115)$$

where $\gamma_{rs}^{TE} a$ is given by (3.58). In (3.115), $j(F^{TE} + \hat{G}_q^{TE} \hat{D}_n^{TE})$ is given by the right-hand side of (3.93) with δ replaced by TE, that is, with g (which appears in the z's of (3.95)–(3.98)) given by

$$g = \gamma_{rs}^{TE}, \quad (3.116)$$

with c_n given by

$$c_n = \frac{2\gamma_{rs}^{TE} c}{(n\pi)^2 + (\gamma_{rs}^{TE} c)^2}, \quad (3.117)$$

and with c_q given by the right-hand side of (3.117) with n replaced by q. Throughout Section 3.2.4, g and c_n are given by (3.116) and (3.117). This g and this c_n are not to be confused with the g and the c_n given by (3.89) and (3.90) of Section 3.2.2.

Expression (3.68) is

$$\hat{z}^{(3)} = F^{(3)} + \hat{G}_q^{TE} \hat{D}_n^{(3)}. \quad (3.118)$$

In (3.118), $F^{(3)}$ is given by [1, eqs. (F.111)-(F.113)]

$$F^{(3)} = \frac{n\pi}{gc}\left\{\begin{array}{ll} (-1)^n c_n c_q \sinh(gc) & , \quad q \neq n \\ (-1)^n c_n c_q \sinh(gc) - c_q & , \quad q = n \neq 0 \\ \dfrac{4\{\sinh(gc) - gc\}}{(gc)^2} & , \quad q = n = 0 \end{array}\right\}. \tag{3.119}$$

As given by (3.119), $F^{(3)} = 0$ when $q = n = 0$. The right-hand side of (3.119) was written so as to be similar to the right-hand side of (3.88). In (3.118), $\hat{G}_q^{TE}$ is given by the right-hand side of (3.91) with g given by (3.116) rather than by (3.89). As for $\hat{D}_n^{(3)}$, in view of (3.77), substitution of [1, eqs. (F.102) and (F.103)] into (3.73) gives

$$\hat{D}_n^{(3)} = \frac{2n\pi c_n}{gc} e^{-gL_3}\left\{\begin{array}{ll} -\sinh(gc/2) & , \quad n \text{ even} \\ \cosh(gc/2) & , \quad n \text{ odd} \end{array}\right\}. \tag{3.120}$$

Comparing the quantities $F^{(3)}$, $\hat{G}_q^{TE}$, and $\hat{D}_n^{(3)}$ of this paragraph with the quantities F^{δ} of (3.88), $\hat{G}_q^{\delta}$ of (3.91), and $\hat{D}_n^{\delta}$ of (3.92) and noting that the latter quantities combined to give (3.93), we obtain

$$F^{(3)} + \hat{G}_q^{TE}\hat{D}_n^{(3)} = -\frac{n\pi}{gc}\left\{j(F^{TE} + \hat{G}_q^{TE}\hat{D}_n^{TE})\right\} \tag{3.121}$$

where $j(F^{TE} + \hat{G}_q^{TE}\hat{D}_n^{TE})$ is given by the right-hand side of (3.93) with δ replaced by TE as described in the third sentence of Section 3.2.4. Substitution of (3.121) into (3.118) gives

$$z^{(3)} = -\frac{n\pi}{gc}\left\{j(F^{TE} + \hat{G}_q^{TE}\hat{D}_n^{TE})\right\}. \tag{3.122}$$

Expression (3.69) is

$$z^{(4)} = -(F^{(4)} + \hat{G}_q^{(4)}\hat{D}_n^{TE}). \tag{3.123}$$

In (3.123), $F^{(4)}$ is given by [1, eqs. (F.129)-(F.131)]

$$F^{(4)} = -\frac{q\pi}{gc}\left\{\begin{array}{ll} (-1)^n c_n c_q \sinh(gc) & , \quad q \neq n \\ (-1)^n c_n c_q \sinh(gc) - c_q & , \quad q = n \neq 0 \\ \dfrac{4\{\sinh(gc) - gc\}}{(gc)^2} & , \quad q = n = 0 \end{array}\right\}. \tag{3.124}$$

As given by (3.124), $F^{(4)} = 0$ when $q = n = 0$. The right-hand side of (3.124) was written so as to be similar to the right-hand side of (3.88). As for $\hat{G}_q^{(4)}$ in (3.123), substitution of [1, eq. (F.126)] into (3.74) gives

$$\hat{G}_q^{(4)} = j\frac{q\pi c_q}{gc}\left\{\sinh(gL_3^+) - (-1)^q \sinh(g(L_3^+ - c))\right\}. \tag{3.125}$$

Substituting (3.77) into (3.125) and using [2, formulas 651.06 and 651.07], we obtain

$$\hat{G}_q^{(4)} = j\frac{2q\pi c_q}{gc}\left\{\begin{array}{ll} \sinh(gc/2)\cosh(gL_3) & , \ q \text{ even} \\ \sinh(gL_3)\cosh(gc/2) & , \ q \text{ odd} \end{array}\right\}. \tag{3.126}$$

In (3.123), $\hat{D}_n^{TE}$ is given by (3.92) with δ replaced by TE. Thus,

$$\hat{D}_n^{TE} = 2jc_n e^{-gL_3}\left\{\begin{array}{ll} -\sinh(gc/2) & , \ n \text{ even} \\ \cosh(gc/2) & , \ n \text{ odd} \end{array}\right. \tag{3.127}$$

where g and c_n are given by (3.116) and (3.117). Comparing the quantities $F^{(4)}$ of (3.124), $\hat{G}_q^{(4)}$ of (3.126), and $\hat{D}_n^{TE}$ of (3.127) with the quantities F^δ of (3.88), $\hat{G}_q^\delta$ of (3.91), and $\hat{D}_n^\delta$ of (3.92) and noting that the latter quantities combined to give (3.93), we obtain

$$F^{(4)} + \hat{G}_q^{(4)}\hat{D}_n^{TE} = \frac{q\pi}{gc}\left\{j(F^{TE} + \hat{G}_q^{TE}\hat{D}_n^{TE})\right\} \tag{3.128}$$

where $j(F^{TE} + \hat{G}_q^{TE}\hat{D}_n^{TE})$ is as in (3.121). Substitution of (3.128) into (3.123) gives

$$\hat{z}^{(4)} = -\frac{q\pi}{gc}\left\{j(F^{TE} + \hat{G}_q^{TE}\hat{D}_n^{TE})\right\}. \tag{3.129}$$

Substitution of (3.116) and [1, eq. (F.121)] into (3.70) gives

$$\hat{z}^{(5)} = \frac{2a}{cx_{rs}'^2}\left\{\begin{array}{ll} 1 & , \ n = q \neq 0 \\ 0 & , \ \text{otherwise} \end{array}\right\} + \frac{j(F^{(5)} + \hat{G}_q^{(4)}\hat{D}_n^{(3)})}{ga} \tag{3.130}$$

where $F^{(5)}$ is given by [1, eqs. (F.132)-(F.134)]

$$F^{(5)} = j\frac{nq\pi^2}{(gc)^2}\left\{\begin{array}{ll} (-1)^n c_n c_q \sinh(gc) & , \ q \neq n \\ (-1)^n c_n c_q \sinh(gc) & , \ q = n \neq 0 \\ \dfrac{4\{\sinh(gc) - gc\}}{(gc)^2} & , \ q = n = 0 \end{array}\right\} + jc_q\left\{\begin{array}{ll} 1 , & q = n \neq 0 \\ 0 , & \text{otherwise} \end{array}\right\}. \tag{3.131}$$

Note that [1, eq. (F.132)] is not correct. Please correct [1, eq. (F.132)] by multiplying the denominator of the right-hand side by $(n\pi)^2 + (\gamma_{rs}^{TE} c)^2$. As given by (3.131), $F^{(5)} = 0$ when $q = n = 0$. The right-hand side of (3.131) was written so as to be as much as possible like the right-hand side of (3.88). In (3.130), $\hat{G}_q^{(4)}$ and $\hat{D}_n^{(3)}$ are given by (3.126) and (3.120), respectively. Comparing the quantities $F^{(5)}$ of (3.131), $\hat{G}_q^{(4)}$ of (3.126), and $\hat{D}_n^{(3)}$ of (3.120) with the quantities F^δ of (3.88), $\hat{G}_q^\delta$ of (3.91), and $\hat{D}_n^\delta$ of (3.92) and noting that the latter quantities combined to give (3.93), we obtain

$$j(F^{(5)} + \hat{G}_q^{(4)} \hat{D}_n^{(3)}) = \frac{nq\pi^2 c_{nq}}{(gc)^2} - c_n \left\{ \begin{array}{ll} 1 \, , & q = n \neq 0 \\ 0 \, , & \text{otherwise} \end{array} \right\} . \tag{3.132}$$

where c_n is given by (3.117) and c_{nq} is given by (3.94) in which the z's are given by (3.95)-(3.98) with g given by (3.116). Substitution of (3.132) into (3.130) gives

$$\hat{z}^{(5)} = \frac{nq\pi^2 c_{nq}}{(ga)(gc)^2} + \left(\frac{2a}{cx_{rs}'^2} - \frac{c_n}{ga} \right) \left\{ \begin{array}{ll} 1 \, , & q = n \neq 0 \\ 0 \, , & \text{otherwise} \end{array} \right\} . \tag{3.133}$$

Chapter 4

The Excitation Vector

The elements of the excitation vector are given by [1, eqs. (5.15) and (5.16)]

$$I_i^{\alpha TM} = \frac{8\phi_o n}{k_{mn} a}\sqrt{\frac{a^2\pi}{bc}}\, y_{sm}\, z_{ccn}\, e^{-j\beta_{01}^{TM} L_3} \tag{4.1}$$

$$I_i^{\alpha TE} = \frac{8\phi_o m}{k_{mn} a}\left(\frac{c}{b}\right)\sqrt{\frac{a^2\pi\epsilon_m\epsilon_n}{4bc}}\, y_{sm}\, z_{ccn}\, e^{-j\beta_{01}^{TM} L_3} \tag{4.2}$$

where [1, eq. (5.21)]

$$y_{sm} = \begin{cases} 0 & , \quad m \text{ even} \\ \frac{2}{m\pi} & , \quad m \text{ odd} \end{cases} \tag{4.3}$$

and [1, eq. (5.24)]

$$\begin{aligned} z_{ccn} = {} & \frac{\sin(n\pi - \beta_{01}^{TM}c)\cos(\beta_{01}^{TM}L_3^+) - 2\sin^2(\frac{n\pi-\beta_{01}^{TM}c}{2})\sin(\beta_{01}^{TM}L_3^+)}{2(n\pi - \beta_{01}^{TM}c)} \\ & + \frac{\sin(n\pi + \beta_{01}^{TM}c)\cos(\beta_{01}^{TM}L_3^+) + 2\sin^2(\frac{n\pi+\beta_{01}^{TM}c}{2})\sin(\beta_{01}^{TM}L_3^+)}{2(n\pi + \beta_{01}^{TM}c)}. \end{aligned} \tag{4.4}$$

In (4.1) and (4.2), α is either 1 or 2, and i determines m and n as described in Appendix A.

Comparing the right-hand side of (4.4) with that of (3.76), we see that

$$z_{ccn} = \frac{1}{2}\left[\hat{G}_n^{TM}\right]_{01} \tag{4.5}$$

where $\left[\hat{G}_n^{TM}\right]_{01}$ is $\hat{G}_n^{TM}$ when $r = 0$ and $s = 1$. Substitution of (4.3) and (4.5) into (4.1) and (4.2) gives

$$I_i^{\alpha TM} = \frac{8\phi_o n}{mk_{mn}b}\sqrt{\frac{b}{\pi c}}\left[\hat{G}_n^{TM}\right]_{01} e^{-j\beta_{01}^{TM}L_3}\left\{\begin{array}{ll} 0, & m \text{ even} \\ 1, & m \text{ odd} \end{array}\right\} \tag{4.6}$$

$$I_i^{\alpha TE} = \frac{8\phi_o}{k_{mn}b}\sqrt{\frac{\epsilon_m\epsilon_n c}{4\pi b}}\left[\hat{G}_n^{TM}\right]_{01} e^{-j\beta_{01}^{TM}L_3}\left\{\begin{array}{ll} 0, & m \text{ even} \\ 1, & m \text{ odd} \end{array}\right\}. \tag{4.7}$$

Multplying both (4.6) and (4.7) by $-je^{j\beta_{01}^{TM}L_3}$ and noting that $\epsilon_m/2 = 1$ whenever m is odd, we obtain

$$-jI_i^{\alpha TM}e^{j\beta_{01}^{TM}L_3} = -j\frac{8\phi_o n}{mk_{mn}b}\sqrt{\frac{b}{\pi c}}\left[\hat{G}_n^{TM}\right]_{01}\left\{\begin{array}{ll} 0, & m \text{ even} \\ 1, & m \text{ odd} \end{array}\right\} \tag{4.8}$$

$$-jI_i^{\alpha TE}e^{j\beta_{01}^{TM}L_3} = -j\frac{8\phi_o}{k_{mn}b}\sqrt{\frac{\epsilon_n c}{2\pi b}}\left[\hat{G}_n^{TM}\right]_{01}\left\{\begin{array}{ll} 0, & m \text{ even} \\ 1, & m \text{ odd} \end{array}\right\}. \tag{4.9}$$

Chapter 5

The Electric Field in the Rectangular Waveguides

In this chapter, modal expansions are found for the electric fields in the rectangular waveguides. Afterwards, the time-average powers of the TE_{10} modes in the rectangular waveguides are obtained.

5.1 Expansions in Terms of the Fields of the Magnetic Currents

The electric field $\underline{E}^{(1)}$ in the left-hand rectangular waveguide (region 1) in Fig. 1.2 is due to $\underline{M}^{(1)}$ and is given by [1, eqs. (2.3) and (2.11)]

$$\underline{E}^{(1)} = \sum_{q=1}\sum_{p=1} V_{pq}^{1TM} \underline{E}^{(1)}(\underline{0}, \underline{M}_{pq}^{1TM}) + \sum_{q=0} \sum_{\substack{p=0 \\ p+q\neq 0}} V_{pq}^{1TE} \underline{E}^{(1)}(\underline{0}, \underline{M}_{pq}^{1TE}). \quad (5.1)$$

In obtaining (5.1), the upper limits on p and q in [1, eq. (2.11)] were suppressed. We truncated the double summations in (5.1) by retaining only terms for which both p and q are so small that, according to (A.2),

$$\sqrt{(p\pi)^2 + \left(\frac{q\pi b}{c}\right)^2} \leq \text{BKM}. \quad (5.2)$$

The electric field $\underline{E}^{(2)}$ in the right-hand rectangular waveguide (region 2) in Fig. 1.2] is similarly given by

$$\underline{E}^{(2)} = \sum_{q=1}\sum_{p=1} V_{pq}^{2TM}\underline{E}^{(2)}(\underline{0},\underline{M}_{pq}^{2TM}) + \sum_{q=0}\sum_{\substack{p=0\\p+q\neq 0}} V_{pq}^{2TE}\underline{E}^{(2)}(\underline{0},\underline{M}_{pq}^{2TE}). \quad (5.3)$$

We approximate the $\underline{M}$'s on the right-hand sides of (5.1) and (5.3) by the $\hat{\underline{M}}$'s given by [1, eqs. (3.3) and 3.13)]. In Sections 5.2 to 5.5, we express the resulting approximate $\underline{E}$'s on the right-hand sides of (5.1) and (5.3) in terms of the modes of the rectangular waveguides.

5.2 The Electric Field of the Magnetic Current $\hat{\underline{M}}_{pq}^{1TM}$

The electric field $\underline{E}^{(1)}(\underline{0},\hat{\underline{M}}_{pq}^{1TM})$ due to $\hat{\underline{M}}_{pq}^{1TM}$ in region 1 of Fig. 1.2 is given by [1, eq. (3.30)]

$$\underline{E}^{(1)}(\underline{0},\hat{\underline{M}}_{pq}^{1TM}) = \{\underline{e}_{pq}^{TM}(y^+,z^+) - \underline{u}_x\frac{k_{pq}^2\psi_{pq}^{TM}(y^+,z^+)}{\gamma_{pq}}\}e^{\gamma_{pq}(x+x_o)}. \quad (5.4)$$

From [1, eqs. (A.3) and (A.13)], the mode field $\underline{E}_{pq}^{TM-}$ is given by

$$\underline{E}_{pq}^{TM-} = -Z_{pq}^{TM}\{\underline{e}_{pq}^{TM}(y^+,z^+) - \underline{u}_x\frac{k_{pq}^2\psi_{pq}^{TM}(y^+,z^+)}{\gamma_{pq}}\}e^{\gamma_{pq}x}. \quad (5.5)$$

In view of (5.5), we recast (5.4) as

$$\underline{E}^{(1)}(\underline{0},\hat{\underline{M}}_{pq}^{1TM}) = \left(-\frac{1}{Z_{pq}^{TM}}\underline{E}_{pq}^{TM-}\right)e^{\gamma_{pq}x_o}. \quad (5.6)$$

5.3 The Electric Field of the Magnetic Current $\hat{\underline{M}}_{pq}^{2TM}$

The electric field $\underline{E}^{(2)}(\underline{0},\hat{\underline{M}}_{pq}^{2TM})$ due to $\hat{\underline{M}}_{pq}^{2TM}$ in region 2 of Fig. 1.2 is given by [1, eq. (3.36)]

$$\underline{E}^{(2)}(\underline{0},\hat{\underline{M}}_{pq}^{2TM}) = \{\underline{e}_{pq}^{TM}(y^+,z^+) + \underline{u}_x\frac{k_{pq}^2\psi_{pq}^{TM}(y^+,z^+)}{\gamma_{pq}}\}e^{-\gamma_{pq}(x-x_o)}. \quad (5.7)$$

From [1, eqs. (A.2) and (A.3)], the mode field $\underline{E}_{pq}^{TM+}$ is given by

$$\underline{E}_{pq}^{TM+} = Z_{pq}^{TM}\{\underline{e}_{pq}^{TM}(y^+, z^+) + \underline{u}_x \frac{k_{pq}^2 \psi_{pq}^{TM}(y^+, z^+)}{\gamma_{pq}}\} e^{-\gamma_{pq} x}. \tag{5.8}$$

In view of (5.8), we recast (5.7) as

$$\underline{E}^{(2)}(\underline{0}, \hat{\underline{M}}_{pq}^{2TM}) = (\frac{1}{Z_{pq}^{TM}} \underline{E}_{pq}^{TM+}) e^{\gamma_{pq} x_o}. \tag{5.9}$$

5.4 The Electric field of the Magnetic Current $\hat{\underline{M}}_{pq}^{1TE}$

The electric field $\underline{E}^{(1)}(\underline{0}, \hat{\underline{M}}_{pq}^{1TE})$ due to $\hat{\underline{M}}_{pq}^{1TE}$ in region 1 of Fig. 1.2 is given by [1, eqs. (3.32) and (3.34)]

$$\begin{aligned} \underline{E}^{(1)}(\underline{0}, \hat{\underline{M}}_{10}^{1TE}) &= \underline{e}_{10}^{TE}(y^+, z^+) \\ &\quad \cdot \frac{j \sin(\beta_{10}(L_1 + x)) + Z_1 Y_{10}^{TE} \cos(\beta_{10}(L_1 + x))}{j \sin(\beta_{10} x_1) + Z_1 Y_{10}^{TE} \cos(\beta_{10} x_1)} \end{aligned} \tag{5.10}$$

$$\underline{E}^{(1)}(\underline{0}, \hat{\underline{M}}_{pq}^{1TE}) = \underline{e}_{pq}^{TE}(y^+, z^+) e^{\gamma_{pq}(x + x_o)}, \quad (p, q) \neq (1, 0) \tag{5.11}$$

where

$$x_1 = L_1 - x_o. \tag{5.12}$$

In obtaining (5.10), we substituted $j\beta_{10}$ for γ_{10} and used [2, formulas 654.6 and 654.7]. Equation (5.10) is recast as

$$\begin{aligned} \underline{E}^{(1)}(\underline{0}, \hat{\underline{M}}_{10}^{1TE}) = & \frac{(Z_1 Y_{10}^{TE} + 1) e^{j\beta_{10}(L_1 + x)} + (Z_1 Y_{10}^{TE} - 1) e^{-j\beta_{10}(L_1 + x)}}{2(j \sin(\beta_{10} x_1) + Z_1 Y_{10}^{TE} \cos(\beta_{10} x_1))} \\ & \cdot \underline{e}_{10}^{TE}(y^+, z^+). \end{aligned} \tag{5.13}$$

From [1, eqs. (A.14) and (A.15)], the mode fields $\underline{E}_{10}^{TE+}$ and $\underline{E}_{10}^{TE-}$ are given by

$$\underline{E}_{10}^{TE+} = \underline{e}_{10}^{TE}(y^+, z^+) e^{-j\beta_{10} x} \tag{5.14}$$

$$\underline{E}_{10}^{TE-} = \underline{e}_{10}^{TE}(y^+, z^+) e^{j\beta_{10} x}, \quad (p, q) \neq (1, 0). \tag{5.15}$$

In obtaining (5.14) and (5.15), we substituted $j\beta_1$ for γ_{10}. In view of (5.14) and (5.15), we recast (5.13) as

$$\underline{E}^{(1)}(\underline{0}, \underline{\hat{M}}_{10}^{1TE}) = \frac{(Z_1 Y_{10}^{TE} + 1)\underline{E}_{10}^{TE-} e^{j\beta_{10} L_1} + (Z_1 Y_{10}^{TE} - 1)\underline{E}_{10}^{TE+} e^{-j\beta_{10} L_1}}{2(j \sin(\beta_{10} x_1) + Z_1 Y_{10}^{TE} \cos(\beta_{10} x_1))}. \tag{5.16}$$

From [1, eq. (A.15)], the mode field $\underline{E}_{pq}^{TE-}$ is given by

$$\underline{E}_{pq}^{TE-} = \underline{e}_{pq}^{TE}(y^+, z^+) e^{\gamma_{pq} x}. \tag{5.17}$$

In view of (5.17), we recast (5.11) as

$$\underline{E}^{(1)}(\underline{0}, \underline{\hat{M}}_{pq}^{1TE}) = \underline{E}_{pq}^{TE-} e^{\gamma_{pq} x_o}, \quad (p, q) \neq (1, 0). \tag{5.18}$$

5.5 The Electric Field of the Magnetic Current $\underline{\hat{M}}_{pq}^{2TE}$

The electric field $\underline{E}^{(2)}(\underline{0}, \underline{\hat{M}}_{pq}^{2TE})$ due to $\underline{\hat{M}}_{pq}^{2TE}$ in region 2 of Fig. 1.2 is given by [1, eqs. (3.38) and (3.40)]

$$\underline{E}^{(2)}(\underline{0}, \underline{\hat{M}}_{10}^{2TE}) = \frac{j \sin(\beta_{10}(L_2 - x)) + Z_2 Y_{10}^{TE} \cos(\beta_{10}(L_2 - x))}{j \sin(\beta_{10} x_2) + Z_2 Y_{10}^{TE} \cos(\beta_{10} x_2)} \underline{e}_{10}^{TE}(y^+, z^+) \tag{5.19}$$

$$\underline{E}^{(2)}(\underline{0}, \underline{\hat{M}}_{pq}^{2TE}) = \underline{e}_{pq}^{TE}(y^+, z^+) e^{-\gamma_{pq}(x - x_o)}, \quad (p, q) \neq (1, 0) \tag{5.20}$$

where

$$x_2 = L_2 - x_o. \tag{5.21}$$

In obtaining (5.19), we substituted $j\beta_{10}$ for γ_{10} and used [2, formulas 654.6 and 654.7]. Equation (5.19) is recast as

$$\underline{E}^{(2)}(\underline{0}, \underline{\hat{M}}_{10}^{2TE}) = \frac{(Z_2 Y_{10}^{TE} + 1)e^{-j\beta_{10}(L_2 - x)} + (Z_2 Y_{10}^{TE} - 1)e^{-j\beta_{10}(L_2 - x)}}{2(j \sin(\beta_{10} x_2) + Z_2 Y_{10}^{TE} \cos(\beta_{10} x_2))}. \tag{5.22}$$

In view of (5.14) and (5.15), we recast (5.22) as

$$\underline{E}^{(2)}(\underline{0},\hat{\underline{M}}_{10}^{2TE}) = \frac{(Z_2Y_{10}^{TE}+1)\underline{E}_{10}^{TE+}e^{j\beta_{10}L_2}+(Z_2Y_{10}^{TE}-1)\underline{E}_{10}^{TE-}e^{-j\beta_{10}L_2}}{2(j\sin(\beta_{10}x_2)+Z_2Y_{10}^{TE}\cos(\beta_{10}x_2))}. \tag{5.23}$$

From [1, eq. (A.14)], the mode field $\underline{E}_{pq}^{TE+}$ is given by

$$\underline{E}_{pq}^{TE+} = \underline{e}_{pq}^{TE}(y^+,z^+)e^{-\gamma_{pq}x}. \tag{5.24}$$

In view of (5.24), we recast (5.20) as

$$\underline{E}^{(2)}(\underline{0},\hat{\underline{M}}_{pq}^{2TE}) = \underline{E}_{pq}^{TE+}e^{\gamma_{pq}x_o}, \quad (p,q)\neq(1,0). \tag{5.25}$$

5.6 Expansions in Terms of Waveguide Modes

Substitution of (5.6), (5.16) and (5.18) into (5.1) with the $\underline{M}$'s replaced by $\hat{\underline{M}}$'s gives

$$\begin{aligned}\underline{E}^{(1)} = {} & \frac{(Z_1Y_{10}^{TE}+1)\underline{E}_{10}^{TE-}e^{j\beta_{10}L_1}+(Z_1Y_{10}^{TE}-1)\underline{E}_{10}^{TE+}e^{-j\beta_{10}L_1}}{2(j\sin(\beta_{10}x_1)+Z_1Y_{10}^{TE}\cos(\beta_{10}x_1))}V_{10}^{1TE} \\ & +\sum_{q=1}\sum_{p=1}V_{pq}^{1TM}\left(\frac{-1}{Z_{pq}^{TM}}\underline{E}_{pq}^{TM-}\right)e^{\gamma_{pq}x_o} \\ & +\sum_{q=0}\sum_{\substack{p=0\\ p+q\neq 0\\ (p,q)\neq(1,0)}}V_{pq}^{1TE}\underline{E}_{pq}^{TE-}e^{\gamma_{pq}x_o}.\end{aligned} \tag{5.26}$$

Substitution of (5.9), (5.23) and (5.25) into (5.3) with the $\underline{M}$'s replaced by $\hat{\underline{M}}$'s gives

$$\begin{aligned}\underline{E}^{(2)} = {} & \frac{(Z_2Y_{10}^{TE}+1)\underline{E}_{10}^{TE+}e^{j\beta_{10}L_2}+(Z_2Y_{10}^{TE}-1)\underline{E}_{10}^{TE-}e^{-j\beta_{10}L_2}}{2(j\sin(\beta_{10}x_2)+Z_2Y_{10}^{TE}\cos(\beta_{10}x_2))}V_{10}^{2TE} \\ & +\sum_{q=1}\sum_{p=1}V_{pq}^{2TM}\left(\frac{1}{Z_{pq}^{TM}}\underline{E}_{pq}^{TM+}\right)e^{\gamma_{pq}x_o} \\ & +\sum_{q=0}\sum_{\substack{p=0\\ p+q\neq 0\\ (p,q)\neq(1,0)}}V_{pq}^{2TE}\underline{E}_{pq}^{TE+}e^{\gamma_{pq}x_o}.\end{aligned} \tag{5.27}$$

5.6.1 Normalization of $\underline{E}^{(1)}$ and $\underline{E}^{(2)}$

The quantities $\underline{E}^{(1)}$ and $\underline{E}^{(2)}$ are due to the z traveling wave whose electric field in the circular waveguide is $\underline{E}_{01}^{TMe+}$ given by

$$\underline{E}_{01}^{TMe+} = Z_{01}^{TMeo}\underline{e}_{01}^{TMe}(\rho,\phi)\underline{e}^{-j\beta_{01}^{TM}z} + \underline{u}_z\frac{(k_{01}^{TM})^2\psi_{01}^{TMe}(\rho,\phi)e^{-j\beta_{01}^{TM}z}}{j\omega\epsilon} \tag{5.28}$$

where

$$Z_{01}^{TMeo} = \frac{\eta\beta_{01}^{TM}}{k}. \tag{5.29}$$

Equation (5.28) was obtained by substituting (3.57) into [1, eq. (B.1)]. Equation (5.29) was obtained by substituting (3.57) into [1, eq. (B.25)]. In this subsection, suitable expressions are found for the quantities $\underline{E}^{(1)}e^{j\beta_{01}^{TM}L_3}/\sqrt{Z_{01}^{TMeo}}$ and $\underline{E}^{(2)}e^{j\beta_{01}^{TM}L_3}/\sqrt{Z_{01}^{TMeo}}$. These quantities are, according to (5.28), due to the z-traveling wave whose transverse electric field is $\sqrt{Z_{01}^{TMeo}}\underline{e}_{01}^{TMe}(\rho,\phi)$ at $z = L_3$ in the circular waveguide. In Section 6.3, it will be shown that the z-directed time-average power associated with this field is unity.

Multiplying both sides of (5.26) and (5.27) by $e^{j\beta_{01}^{TM}L_3}/\sqrt{Z_{01}^{TMeo}}$ and using (5.29), we obtain

$$\begin{aligned}\left(\frac{e^{j\beta_{01}^{TM}L_3}}{\sqrt{Z_{01}^{TMeo}}}\right)\underline{E}^{(1)} &= \frac{C_{10}^{1TE-}e^{j\beta_{10}x_o}\underline{E}_{10}^{TE-} + C_{10}^{1TE+}e^{-j\beta_{10}x_o}\underline{E}_{10}^{TE+}}{\sqrt{Y_{10}^{TE}}} \\ &- \sum_{q=1}\sum_{p=1}\frac{C_{pq}^{1TM-}e^{\gamma_{pq}x_o}\underline{E}_{pq}^{TM-}}{\sqrt{|Z_{pq}^{TM}|}} + \sum_{q=0}\sum_{\substack{p=0\\p+q\neq 0\\(p,q)\neq(1,0)}}\frac{C_{pq}^{1TE-}e^{\gamma_{pq}x_o}\underline{E}_{pq}^{TE-}}{\sqrt{|Y_{pq}^{TE}|}}\end{aligned} \tag{5.30}$$

$$\begin{aligned}\left(\frac{e^{j\beta_{01}^{TM}L_3}}{\sqrt{Z_{01}^{TMeo}}}\right)\underline{E}^{(2)} &= \frac{C_{10}^{2TE+}e^{j\beta_{10}x_o}\underline{E}_{10}^{TE+} + C_{10}^{2TE-}e^{-j\beta_{10}x_o}\underline{E}_{10}^{TE-}}{\sqrt{Y_{10}^{TE}}} \\ &+ \sum_{q=1}\sum_{p=1}\frac{C_{pq}^{2TM+}e^{\gamma_{pq}x_o}\underline{E}_{pq}^{TM+}}{\sqrt{|Z_{pq}^{TM}|}} + \sum_{q=0}\sum_{\substack{p=0\\p+q\neq 0\\(p,q)\neq(1,0)}}\frac{C_{pq}^{2TE+}e^{\gamma_{pq}x_o}\underline{E}_{pq}^{TE+}}{\sqrt{|Y_{pq}^{TE}|}}.\end{aligned} \tag{5.31}$$

$$\cdot \left(\frac{V_{10}^{2TE} e^{j\beta_{01}^{TM} L_3}}{\eta} \right) \tag{5.37}$$

$$C_{pq}^{2TM+} = \frac{jk}{\sqrt{\gamma_{pq}\beta_{01}^{TM}}} \left(\frac{V_{pq}^{2TM} e^{j\beta_{01}^{TM} L_3}}{\eta} \right) \tag{5.38}$$

$$C_{pq}^{2TE+} = \left(\sqrt{\frac{\gamma_{pq}}{\beta_{01}^{TM}}} \right) \left(\frac{V_{pq}^{2TE} e^{j\beta_{01}^{TM} L_3}}{\eta} \right). \tag{5.39}$$

5.7 Time-Average Power

What is the x-directed time-average power associated with an arbitrary electromagnetic field $(\underline{E}, \underline{H})$ in a source-free region of a rectangular waveguide? This field can be expressed as

$$\underline{E} = \sum_{p,q} (C_{pq}^{TM+} \underline{E}_{pq}^{TM+} + C_{pq}^{TM-} \underline{E}_{pq}^{TM-} + C_{pq}^{TE+} \underline{E}_{pq}^{TE+} + C_{pq}^{TE-} \underline{E}_{pq}^{TE-}) \tag{5.40}$$

$$\underline{H} = \sum_{p,q} (C_{pq}^{TM+} \underline{H}_{pq}^{TM+} + C_{pq}^{TM-} \underline{H}_{pq}^{TM-} + C_{pq}^{TE+} \underline{H}_{pq}^{TE+} + C_{pq}^{TE-} \underline{H}_{pq}^{TE-}). \tag{5.41}$$

On the right-hand sides of (5.40) and (5.41), the C's are constants, and the $\underline{E}$'s and the $\underline{H}$'s are the mode fields defined by [1, eqs. (A.2), (A.3), (A.14), and (A.15)]. The x-directed time-average power P associated with $(\underline{E}, \underline{H})$ is given by [4, eqs. (1-57) and (1-58)]

$$P = \int_0^b dy^+ \int_0^c dz^+ \mathrm{Re}(\underline{E} \times \underline{H}^*) \cdot \underline{u}_x \tag{5.42}$$

where "$*$" denotes complex conjugate and "Re" denotes real part. Substituting (5.40) and (5.41) into (5.42) and using the previously mentioned definitions of the mode fields and the last of the orthogonality relations [1, eq. (A.26)], we obtain

$$\begin{aligned} P = & \sum_{p,q} \{ |C_{pq}^{TM+}|^2 e^{-2\mathrm{Re}(\gamma_{pq})x} - |C_{pq}^{TM-}|^2 e^{2\mathrm{Re}(\gamma_{pq})x} \} \mathrm{Re}(Z_{pq}^{TM}) \\ & -2 \sum_{p,q} \mathrm{Imag}\{ C_{pq}^{TM+} (C_{pq}^{TM-})^* e^{-2j\mathrm{Imag}(\gamma_{pq})x} \} \mathrm{Imag}(Z_{pq}^{TM}) \end{aligned}$$

In (5.30),

$$C_{10}^{1TE-} = \left(\sqrt{\frac{\beta_{10}}{\beta_{01}^{TM}}}\right)\left\{\frac{(Z_1 Y_{10}^{TE}+1)e^{j\beta_{10}x_1}}{2(Z_1 Y_{10}^{TE}\cos(\beta_{10}x_1)+j\sin(\beta_{10}x_1))}\right\} \cdot \left(\frac{V_{10}^{1TE}e^{j\beta_{01}^{TM}L_3}}{\eta}\right) \tag{5.32}$$

$$C_{10}^{1TE+} = \left(\sqrt{\frac{\beta_{10}}{\beta_{01}^{TM}}}\right)\left\{\frac{(Z_1 Y_{10}^{TE}-1)e^{-j\beta_{10}x_1}}{2(Z_1 Y_{10}^{TE}\cos(\beta_{10}x_1)+j\sin(\beta_{10}x_1))}\right\} \cdot \left(\frac{V_{10}^{1TE}e^{j\beta_{01}^{TM}L_3}}{\eta}\right) \tag{5.33}$$

$$C_{pq}^{1TM-} = \frac{jk}{\sqrt{\gamma_{pq}\beta_{01}^{TM}}}\left(\frac{V_{pq}^{1TM}e^{j\beta_{01}^{TM}L_3}}{\eta}\right) \tag{5.34}$$

$$C_{pq}^{1TE-} = \left(\sqrt{\frac{\gamma_{pq}}{\beta_{01}^{TM}}}\right)\left(\frac{V_{pq}^{1TE}e^{j\beta_{01}^{TM}L_3}}{\eta}\right). \tag{5.35}$$

In (5.31),

$$C_{10}^{2TE+} = \left(\sqrt{\frac{\beta_{10}}{\beta_{01}^{TM}}}\right)\left\{\frac{(Z_2 Y_{10}^{TE}+1)e^{j\beta_{10}x_2}}{2(Z_2 Y_{10}^{TE}\cos(\beta_{10}x_2)+j\sin(\beta_{10}x_2))}\right\} \cdot \left(\frac{V_{10}^{2TE}e^{j\beta_{01}^{TM}L_3}}{\eta}\right) \tag{5.36}$$

$$C_{10}^{2TE+} = \left(\sqrt{\frac{\beta_{10}}{\beta_{01}^{TM}}}\right)\left\{\frac{(Z_2 Y_{10}^{TE}-1)e^{-j\beta_{10}x_2}}{2(Z_2 Y_{10}^{TE}\cos(\beta_{10}x_2)+j\sin(\beta_{10}x_2))}\right\}$$

$$+\sum_{p,q}\{|C_{pq}^{TE+}|^2 e^{-2\mathrm{Re}(\gamma_{pq})x} - |C_{pq}^{TE-}|^2 e^{2\mathrm{Re}(\gamma_{pq})x}\}\mathrm{Re}(Y_{pq}^{TE})$$
$$-2\sum_{p,q}\mathrm{Imag}\{C_{pq}^{TE+}(C_{pq}^{TE-})^* e^{-2j\mathrm{Imag}(\gamma_{pq})x}\mathrm{Imag}(Y_{pq}^{TE}) \tag{5.43}$$

where "Imag" denotes imaginary part. In (5.43), we have [1, eqs. (A.13) and (A.25)]

$$Z_{pq}^{TM} = -j\frac{\eta\gamma_{pq}}{k} \tag{5.44}$$
$$Y_{pq}^{TE} = -j\frac{\gamma_{pq}}{k\eta} \tag{5.45}$$

where $\gamma_{pq} = -j\beta_{pq}$ if the mode propagates, and γ_{pq} is purely real if the mode does not propagate. Since only the TE_{10} mode propagates, (5.43) reduces to

$$P = \frac{2\eta}{k}\left\{\sum_{p,q}\gamma_{pq}\mathrm{Imag}\{C_{pq}^{TM+}(C_{pq}^{TM-})^*\}\right\} + \frac{\beta_{10}}{k\eta}\{|C_{10}^{TE+}|^2 - |C_{10}^{TE-}|^2\}$$
$$+\frac{2}{k\eta}\left\{\sum_{\substack{p,q\\(p,q)\neq(1,0)}}\gamma_{pq}\mathrm{Imag}\{C_{pq}^{TE+}(C_{pq}^{TE-})^*\}\right\}. \tag{5.46}$$

5.7.1 Time-Average Power in the Rectangular Waveguides

The normalized electric fields $\underline{E}^{(1)}e^{j\beta_{01}^{TM}L_3}/\sqrt{Z_{01}^{TMeo}}$ and $\underline{E}^{(2)}e^{j\beta_{01}^{TM}L_3}/\sqrt{Z_{01}^{TMeo}}$ of (5.37) and (5.38) are due to the z-traveling wave whose transverse electric field is $\underline{E}_{01}^{TMe+}e^{-j\beta_{01}^{TM}(z-L_3)}/\sqrt{Z_{01}^{TMeo}}$ in the circular waveguide. The z-directed time-average power of this field in the circular waveguide is, as given by an expression very similar to (5.31), equal to unity. The $-x$-directed time-average power of $\underline{E}^{(1)}e^{j\beta_{01}^{TM}L_3}/\sqrt{Z_{01}^{TMeo}}$ is $P^{(1)}/Z_{01}^{TMeo}$ given by (5.31) as

$$\frac{P^{(1)}}{Z_{01}^{TMeo}} = |C_{10}^{1TE-}|^2 - |C_{10}^{1TE+}|^2. \tag{5.47}$$

The x-directed time-average power of $\underline{E}^{(2)} e^{j\beta_{01}^{TM} L_3} / \sqrt{Z_{01}^{TMeo}}$ is $P^{(2)}/Z_{01}^{TMeo}$ given by (5.31) as

$$\frac{P^{(2)}}{Z_{01}^{TEeo}} = |C_{10}^{2TE+}|^2 - |C_{10}^{2TE-}|^2. \tag{5.48}$$

When the incident time-average power in the circular waveguide is unity, the time-average power P_t transmitted into the rectangular waveguides is the sum of (5.47) and (5.48):

$$P_t = |C_{10}^{1TE-}|^2 - |C_{10}^{1TE+}|^2 + |C_{10}^{2TE+}|^2 - |C_{10}^{2TE-}|^2. \tag{5.49}$$

Chapter 6

The Electric Field in the Circular Waveguide

In this chapter, a modal expansion is found for the electric field $\underline{E}^{(3)}$ in the part of the circular waveguide for which $z_s < z < -c/2$ where z_s is such that the impressed source $\underline{J}^{\text{imp}}$ lies in the region for which $z < z_s$. See Fig. 1.2 where the entire region inside the circular waveguide is called Region 3. As stated in Chapter 1, only the TM_{01} and TE_{11} modes propagate in Region 3. The coefficients of the TM_{01} and TE_{11} modes in the modal expansion for $\underline{E}^{(3)}$ are then expressed in forms suitable for computation. Finally, the time-average powers of the TM_{01} and TE_{11} modal contributions to $\underline{E}^{(3)}$ are obtained.

The electric field $\underline{E}^{(3)}$ in the circular waveguide is given by [1, eq. (2.7)]

$$\underline{E}^{(3)} = \underline{E}^{(3)}(\underline{0}, -\underline{M}^{(1)} - \underline{M}^{(2)}) + \underline{E}^{(3)}(\underline{J}^{\text{imp}}, \underline{0}) \tag{6.1}$$

where $\underline{E}^{(3)}(\underline{J}^{\text{imp}}, \underline{0})$ is the electric field due to $\underline{J}^{\text{imp}}$, and $\underline{E}^{(3)}(\underline{0}, -\underline{M}^{(1)} - \underline{M}^{(2)})$ is the electric field due to $-\underline{M}^{(1)} - \underline{M}^{(2)}$ where $\underline{M}^{(1)} - \underline{M}^{(2)}$ is the combination of $-\underline{M}^{(1)}$ and $-\underline{M}^{(2)}$. Each of the sources $\underline{J}^{\text{imp}}$, $-\underline{M}^{(1)}$, and $-\underline{M}^{(2)}$ radiates in the circular waveguide with the apertures closed, with the short at $z = L_3$, and with a matched load at the other end where $z << 0$.

6.1 The Electric Field $\underline{E}^{(3)}(\underline{0}, -\underline{M}^{(1)} - \underline{M}^{(2)})$

Since $\underline{M}^{(1)}$ and $\underline{M}^{(2)}$ are given by [1, eqs. (2.11) and (2.12)], we have

$$\begin{aligned} \underline{E}^{(3)}(\underline{0}, -\underline{M}^{(1)} - \underline{M}^{(2)}) &= -\sum_{\gamma,q,p} V_{pq}^{\gamma TM} \underline{E}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma TM}) \\ &\quad - \sum_{\gamma,q,p} V_{pq}^{\gamma TE} \underline{E}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma TE}). \end{aligned} \tag{6.2}$$

The first $\sum_{\gamma,q,p}$ on the right-hand side of (6.2) stands for $\sum_{\gamma=1}^{2} \sum_{q=1} \sum_{p=1}$. The second $\sum_{\gamma,q,p}$ stands for $\sum_{\gamma=1}^{2} \sum_{q=0} \sum_{\substack{p=0 \\ p+q\neq 0}}$. Here, no upper limits are placed on the indices p and q. Terms are retained only for which p and q are so small that (5.2) is true. The $\underline{E}^{(3)}$'s on the right-hand side of (6.2) are given by [1, eqs. (4.47) and (4.61)]:

$$\begin{aligned} \underline{E}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma TM}) &= -\frac{4a}{k_{pq}\sqrt{bc}} \{ (\frac{q}{c}) \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TM})^2 J_r'(k_{rs}^{TM} a) \underline{E}^{\gamma TM\phi}}{2x_{rs}^2 J_{r+1}^2(x_{rs})} \\ &\quad + (\frac{q}{c}) \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{(k_{rs}^{TE})^2 r J_r(k_{rs}^{TE} a) \underline{E}^{\gamma TE\phi}}{(k_{rs}^{TE} a)(x_{rs}'^2 - r^2) J_r^2(x_{rs}')} \\ &\quad - (\frac{p}{b})(\frac{\sin\phi_o}{\phi_o}) \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TE})^3 J_r(k_{rs}^{TE} a) \underline{E}^{\gamma TEz}}{2\gamma_{rs}^{TE}(x_{rs}'^2 - r^2) J_r^2(x_{rs}')} \} \end{aligned} \tag{6.3}$$

$$\begin{aligned} \underline{E}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma TE}) &= -\frac{4a}{k_{pq}} \sqrt{\frac{\epsilon_p \epsilon_q}{4bc}} \{ (\frac{p}{b}) \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TM})^2 J_r'(k_{rs}^{TM} a) \underline{E}^{\gamma TM\phi}}{2x_{rs}^2 J_{r+1}^2(x_{rs})} \\ &\quad + (\frac{p}{b}) \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{(k_{rs}^{TE})^2 r J_r(k_{rs}^{TE} a) \underline{E}^{\gamma TE\phi}}{(k_{rs}^{TE} a)(x_{rs}'^2 - r^2) J_r^2(x_{rs}')} \\ &\quad + (\frac{q}{c})(\frac{\sin\phi_o}{\phi_o}) \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \frac{\epsilon_r (k_{rs}^{TE})^3 J_r(k_{rs}^{TE} a) \underline{E}^{\gamma TEz}}{2\gamma_{rs}^{TE}(x_{rs}'^2 - r^2) J_r^2(x_{rs}')} \}. \end{aligned} \tag{6.4}$$

In (6.3) and (6.4), γ_{rs}^{TE} is given by (3.58). Furthermore, $\underline{E}^{\gamma TM\phi}$, $\underline{E}^{\gamma TE\phi}$, and $\underline{E}^{\gamma TEz}$ are given by [1, eqs. (4.74)-(4.76)]:

$$\begin{aligned} \underline{E}^{\gamma TM\phi} &= \left\{ -z^{TM1} \sinh(\gamma_{rs}^{TM}(L_3 - z)) + z^{TM2} e^{-\gamma_{rs}^{TM}(L_3 - z)} \right\} \\ &\quad \cdot \left\{ \underline{u}_\rho \phi^{\gamma 2} J_r'(k_{rs}^{TM}\rho) - \underline{u}_\phi \frac{\phi^{\gamma 1} r J_r(k_{rs}^{TM}\rho)}{k_{rs}^{TM}\rho} \right\} \end{aligned}$$

$$+\left\{z^{TM1}\cosh(\gamma_{rs}^{TM}(L_3-z))+z^{TM2}e^{-\gamma_{rs}^{TM}(L_3-z)}\right\}\underline{u}_z\frac{\phi^{\gamma 2}k_{rs}^{TM}J_r(k_{rs}^{TM}\rho)}{\gamma_{rs}^{TM}} \quad (6.5)$$

$$\underline{E}^{\gamma TE\phi} = \left\{-z^{TE1}\sinh(\gamma_{rs}^{TE}(L_3-z))+z^{TE2}e^{-\gamma_{rs}^{TE}(L_3-z)}\right\} \cdot\left\{\underline{u}_\rho\frac{\phi^{\gamma 2}rJ_r(k_{rs}^{TE}\rho)}{k_{rs}^{TE}\rho}-\underline{u}_\phi\phi^{\gamma 1}J_r'(k_{rs}^{TE}\rho)\right\} \quad (6.6)$$

$$\underline{E}^{\gamma TEz} = \left\{z^{TE3}\sinh(\gamma_{rs}^{TE}(L_3-z))+z^{TE4}e^{-\gamma_{rs}^{TE}(L_3-z)}\right\} \cdot(-1)^\gamma\left\{\underline{u}_\rho\frac{\phi^{\gamma 3}rJ_r(k_{rs}^{TE}\rho)}{k_{rs}^{TE}\rho}+\underline{u}_\phi\phi^{\gamma 4}J_r'(k_{rs}^{TE}\rho)\right\}. \quad (6.7)$$

The ϕ's in (6.5)-(6.7) are given by [1, eqs. (E.10)-(E.13)]:

$$\phi^{\gamma 1} = -(-1)^\gamma\left\{\phi_p^{(1)}\cos(\frac{ry^{\gamma+}}{x_o})-\phi_p^{(2)}\sin(\frac{ry^{\gamma+}}{x_o})\right\} \quad (6.8)$$

$$\phi^{\gamma 2} = \phi_p^{(2)}\cos(\frac{ry^{\gamma+}}{x_o})+\phi_p^{(1)}\sin(\frac{ry^{\gamma+}}{x_o}) \quad (6.9)$$

$$\phi^{\gamma 3} = -(-1)^\gamma\left\{\phi_p^{(3)}\cos(\frac{ry^{\gamma+}}{x_o})-\phi_p^{(4)}\sin(\frac{ry^{\gamma+}}{x_o})\right\} \quad (6.10)$$

$$\phi^{\gamma 4} = \phi_p^{(4)}\cos(\frac{ry^{\gamma+}}{x_o})+\phi_p^{(3)}\sin(\frac{ry^{\gamma+}}{x_o}). \quad (6.11)$$

The y's on the right-hand sides of (6.8)-(6.11) are given by [1, eqs. (2.15) and (2.16)]:

$$y^{1+} = (\pi-\phi)x_o+\frac{b}{2} \quad (6.12)$$

$$y^{2+} = \phi x_o+\frac{b}{2}. \quad (6.13)$$

The ϕ_p's on the right-hand sides of (6.8)-(6.11) are given by (3.40)-(3.43). For simplicity, we assume that $z<-c/2$ so that [1, eqs. (4.77) and (4.83)]

$$z^{TM1}=z^{TE1}=z^{TE3}=0. \quad (6.14)$$

Since $z<-c/2$, the remaining superscripted z's in (6.5)-(6.7) (namely z^{TM2}, z^{TE2}, and z^{TE4}) are, as stated in [1, page 34], given by their expressions in

[1, Appendix F] with z^+ replaced by zero. For $x_{rs} < ka$, we have [1, eqs. (F.26) and (F.35)]

$$\frac{z^{\delta 2}}{c} = \left[\left\{\sin(q^{\delta-}c)\cos(\beta^{\delta}_{rs}L_3^+) - 2\sin^2(\frac{q^{\delta-}c}{2})\sin(\beta^{\delta}_{rs}L_3^+)\right\}/(2q^{\delta-}c)\right]$$
$$+\left[\left\{\sin(q^{\delta+}c)\cos(\beta^{\delta}_{rs}L_3^+) + 2\sin^2(\frac{q^{\delta+}c}{2})\sin(\beta^{\delta}_{rs}L_3^+)\right\}/(2q^{\delta+}c)\right] \quad (6.15)$$

$$\frac{z^{TE4}}{c} = -j$$
$$\cdot\left[\left\{\sin(q^{TE-}c)\cos(\beta^{TE}_{rs}L_3^+) - 2\sin^2(\frac{q^{TE-}c}{2})\sin(\beta^{TE}_{rs}L_3^+)\right\}/(2q^{TE-}c)\right]$$
$$+j\left[\left\{\sin(q^{TE+}c)\cos(\beta^{TE}_{rs}L_3^+) + 2\sin^2(\frac{q^{TE+}c}{2})\sin(\beta^{TE}_{rs}L_3^+)\right\}/(2q^{TE+}c)\right]. \quad (6.16)$$

For $x_{rs} \geq ka$, we have [1, eqs. (F.33) and (F.42)]

$$\frac{z^{\delta 2}}{c} = \left\{\gamma^{\delta}_{rs}c\sinh(\gamma^{\delta}_{rs}L_3^+) - (-1)^q\gamma^{\delta}_{rs}c\sinh(\gamma^{\delta}_{rs}(L_3^+ - c))\right\}/\left\{(q\pi)^2 + (\gamma^{\delta}_{rs}c)^2\right\} \quad (6.17)$$

$$\frac{z^{TE4}}{c} = q\pi\left\{\sinh(\gamma^{TE}_{rs}L_3^+) - (-1)^q\sinh(\gamma^{TE}_{rs}(L_3^+ - c))\right\}/\left\{(q\pi)^2 + (\gamma^{TE}_{rs}c)^2\right\}. \quad (6.18)$$

In (6.15) and (6.17), δ is either TM or TE.

6.1.1 The Quantities $\underline{E}^{\gamma TM\phi}$, $\underline{E}^{\gamma TE\phi}$, and $\underline{E}^{\gamma TEz}$

In this section, expressions (6.5)-(6.7) for $\underline{E}^{\gamma TM\phi}$, $\underline{E}^{\gamma TE\phi}$, and $\underline{E}^{\gamma TEz}$ are first reduced by means of (6.14), and then expanded by means of (6.8)-(6.13). Next, the $-z$-traveling modes of the circular waveguide are introduced. Finally, the expressions for $\underline{E}^{\gamma TM\phi}$, $\underline{E}^{\gamma TE\phi}$, and $\underline{E}^{\gamma TEz}$ that were obtained by means of (6.8)-(6.14) are recast in terms of the $-z$-traveling modes of the circular waveguide.

Reduction of (6.5)–(6.7) by means of (6.14)

Substitution of (6.14) into (6.5)–(6.7) gives

$$\begin{aligned}
\underline{E}^{\gamma TM\phi} &= z^{TM2} e^{-\gamma_{rs}^{TM}(L_3-z)} \\
&\cdot \left\{ \underline{u}_\rho \phi^{\gamma 2} J_r'(k_{rs}^{TM}\rho) - \underline{u}_\phi \frac{\phi^{\gamma 1} r J_r(k_{rs}^{TM}\rho)}{k_{rs}^{TM}\rho} + \underline{u}_z \frac{\phi^{\gamma 2} k_{rs}^{TM} J_r(k_{rs}^{TM}\rho)}{\gamma_{rs}^{TM}} \right\} \qquad (6.19) \\
\underline{E}^{\gamma TE\phi} &= z^{TE2} e^{-\gamma_{rs}^{TE}(L_3-z)} \left\{ \underline{u}_\rho \frac{\phi^{\gamma 2} r J_r(k_{rs}^{TE}\rho)}{k_{rs}^{TE}\rho} - \underline{u}_\phi \phi^{\gamma 1} J_r'(k_{rs}^{TE}\rho) \right\} \qquad (6.20) \\
\underline{E}^{\gamma TEz} &= z^{TE4} e^{-\gamma_{rs}^{TE}(L_3-z)} (-1)^\gamma \left\{ \underline{u}_\rho \frac{\phi^{\gamma 3} r J_r(k_{rs}^{TE}\rho)}{k_{rs}^{TE}\rho} + \underline{u}_\phi \phi^{\gamma 4} J_r'(k_{rs}^{TE}\rho) \right\}. \qquad (6.21)
\end{aligned}$$

Expansion of (6.19)–(6.21) by means of (6.8)–(6.13)

Substituting (6.12) and (6.13) into the arguments of the trigonometric functions in (6.8)–(6.11) and using [2, formulas 401.01–401.04], we obtain

$$\cos\left(\frac{ry^{\gamma+}}{x_o}\right) = (-1)^{\gamma r} \left\{ \cos\left(\frac{rb}{2x_o}\right) \cos(r\phi) - (-1)^\gamma \sin\left(\frac{rb}{2x_o}\right) \sin(r\phi) \right\} \qquad (6.22)$$

$$\sin\left(\frac{ry^{\gamma+}}{x_o}\right) = (-1)^{\gamma r} \left\{ \sin\left(\frac{rb}{2x_o}\right) \cos(r\phi) + (-1)^\gamma \cos\left(\frac{rb}{2x_o}\right) \sin(r\phi) \right\} \qquad (6.23)$$

Substitution of (6.22) and (6.23) into (6.8)–(6.11) gives

$$\phi^{\gamma 1} = (-1)^{\gamma r} \left\{ (-1)^{\gamma+1} \phi_p^{b1} \cos(r\phi) + \phi_p^{b2} \sin(r\phi) \right\} \qquad (6.24)$$

$$\phi^{\gamma 2} = (-1)^{\gamma r} \left\{ \phi_p^{b2} \cos(r\phi) + (-1)^\gamma \phi_p^{b1} \sin(r\phi) \right\} \qquad (6.25)$$

$$\phi^{\gamma 3} = (-1)^{\gamma r} \left\{ (-1)^{\gamma+1} \phi_p^{b3} \cos(r\phi) + \phi_p^{b4} \sin(r\phi) \right\} \qquad (6.26)$$

$$\phi^{\gamma 4} = (-1)^{\gamma r} \left\{ \phi_p^{b4} \cos(r\phi) + (-1)^\gamma \phi_p^{b3} \sin(r\phi) \right\} \qquad (6.27)$$

where

$$\phi_p^{b1} = \phi_p^{(1)} \cos\left(\frac{rb}{2x_o}\right) - \phi_p^{(2)} \sin\left(\frac{rb}{2x_o}\right) \qquad (6.28)$$

$$\phi_p^{b2} = \phi_p^{(2)} \cos\left(\frac{rb}{2x_o}\right) + \phi_p^{(1)} \sin\left(\frac{rb}{2x_o}\right) \qquad (6.29)$$

$$\phi_p^{b3} = \phi_p^{(3)} \cos\left(\frac{rb}{2x_o}\right) - \phi_p^{(4)} \sin\left(\frac{rb}{2x_o}\right) \tag{6.30}$$

$$\phi_p^{b4} = \phi_p^{(4)} \cos\left(\frac{rb}{2x_o}\right) + \phi_p^{(3)} \sin\left(\frac{rb}{2x_o}\right) \tag{6.31}$$

Substituting (6.24)–(6.27) into (6.19)–(6.21), we obtain

$$\begin{aligned} \underline{E}^{\gamma TM\phi} &= (-1)^{\gamma r} z^{TM2} e^{-\gamma_{rs}^{TM}(L_3-z)} \Big\{ (-1)^{\gamma} \phi_p^{b1} \\ &\quad \cdot \{ \underline{u}_\rho J_r'(k_{rs}^{TM}\rho) \sin(r\phi) + \underline{u}_\phi \frac{r J_r(k_{rs}^{TM}\rho)\cos(r\phi)}{k_{rs}^{TM}\rho} \\ &\quad + \underline{u}_z \frac{k_{rs}^{TM} J_r(k_{rs}^{TM}\rho)\sin(r\phi)}{\gamma_{rs}^{TM}} \} + \phi_p^{b2} \{ \underline{u}_\rho J_r'(k_{rs}^{TM}\rho)\cos(r\phi) \\ &\quad - \underline{u}_\phi \frac{r J_r(k_{rs}^{TM}\rho)\sin(r\phi)}{k_{rs}^{TM}\rho} + \underline{u}_z \frac{k_{rs}^{TM} J_r(k_{rs}^{TM}\rho)\cos(r\phi)}{\gamma_{rs}^{TM}} \} \Big\} \end{aligned} \tag{6.32}$$

$$\begin{aligned} \underline{E}^{\gamma TE\phi} &= (-1)^{\gamma r} z^{TE2} e^{-\gamma_{rs}^{TE}(L_3-z)} \Big\{ \\ &\quad (-1)^{\gamma} \phi_p^{b1} \{ \underline{u}_\rho \frac{r J_r(k_{rs}^{TE}\rho)\sin(r\phi)}{k_{rs}^{TE}\rho} + \underline{u}_\phi J_r'(k_{rs}^{TE}\rho)\cos(r\phi) \} \\ &\quad + \phi_p^{b2} \{ \underline{u}_\rho \frac{r J_r(k_{rs}^{TE}\rho)\cos(r\phi)}{k_{rs}^{TE}\rho} - \underline{u}_\phi J_r'(k_{rs}^{TE}\rho)\sin(r\phi) \} \Big\} \end{aligned} \tag{6.33}$$

$$\begin{aligned} \underline{E}^{\gamma TEz} &= (-1)^{\gamma r} z^{TE4} e^{-\gamma_{rs}^{TE}(L_3-z)} \Big\{ \\ &\quad -\phi_p^{b3} \{ \underline{u}_\rho \frac{r J_r(k_{rs}^{TE}\rho)\cos(r\phi)}{k_{rs}^{TE}\rho} - \underline{u}_\phi J_r'(k_{rs}^{TE}\rho)\sin(r\phi) \} + (-1)^{\gamma} \\ &\quad \cdot \phi_p^{b4} \{ \underline{u}_\rho \frac{r J_r(k_{rs}^{TE}\rho)\sin(r\phi)}{k_{rs}^{TE}\rho} + \underline{u}_\phi J_r'(k_{rs}^{TE}\rho)\cos(r\phi) \} \Big\}. \end{aligned} \tag{6.34}$$

The $-z$-Traveling Modes of the Circular Waveguide

The modes of the circular waveguide that travel in the $-z$-direction are $\underline{E}_{rs}^{TMe-}$, $\underline{E}_{rs}^{TMo-}$, $\underline{E}_{rs}^{TEe-}$, and $\underline{E}_{rs}^{TEo-}$ given by [1, eqs. (B.2), (B.27), (B.36), and (B.56)]

$$\underline{E}_{rs}^{TMe-} = \left\{ -Z_{rs}^{TMeo} \underline{e}_{rs}^{TMe}(\rho, \phi) + \underline{u}_z \frac{(k_{rs}^{TM})^2 \psi_{rs}^{TMe}(\rho, \phi)}{j\omega\epsilon} \right\} e^{\gamma_{rs}^{TM} z} \quad (6.35)$$

$$\underline{E}_{rs}^{TMo-} = \left\{ -Z_{rs}^{TMeo} \underline{e}_{rs}^{TMo}(\rho, \phi) + \underline{u}_z \frac{(k_{rs}^{TM})^2 \psi_{rs}^{TMe}(\rho, \phi)}{j\omega\epsilon} \right\} e^{\gamma_{rs}^{TM} z} \quad (6.36)$$

$$\underline{E}_{rs}^{TEe-} = \underline{e}_{rs}^{TEe}(\rho, \phi) e^{\gamma_{rs}^{TE} z} \quad (6.37)$$

$$\underline{E}_{rs}^{TEo-} = \underline{e}_{rs}^{TEo}(\rho, \phi) e^{\gamma_{rs}^{TE} z} \quad (6.38)$$

where [1, eqs. (B.25), (B.7), (B.22), (B.30), (B.33), (B.51), and (B.62)]

$$Z_{rs}^{TMeo} = \frac{\gamma_{rs}^{TM}}{j\omega\epsilon} \quad (6.39)$$

$$\psi_{rs}^{TMe}(\rho, \phi) = \sqrt{\frac{\epsilon_r}{\pi}} \frac{J_r(k_{rs}^{TM}\rho)\cos(r\phi)}{x_{rs} J_{r+1}(x_{rs})} \quad (6.40)$$

$$\underline{e}_{rs}^{TMe}(\rho, \phi) = -\sqrt{\frac{\epsilon_r}{\pi}} \left(\frac{1}{a J_{r+1}(x_{rs})} \right) \cdot \left\{ \underline{u}_\rho J_r'(k_{rs}^{TM}\rho)\cos(r\phi) - \underline{u}_\phi \frac{r J_r(k_{rs}^{TM}\rho)\sin(r\phi)}{k_{rs}^{TM}\rho} \right\} \quad (6.41)$$

$$\psi_{rs}^{TMo}(\rho, \phi) = \sqrt{\frac{2}{\pi}} \frac{J_n(k_{np}^{TM}\rho)\sin(n\phi)}{x_{np} J_{n+1}(x_{np})} \quad (6.42)$$

$$\underline{e}_{rs}^{TMo}(\rho, \phi) = -\sqrt{\frac{2}{\pi}} \left(\frac{1}{a J_{r+1}(x_{rs})} \right) \cdot \left\{ \underline{u}_\rho J_r'(k_{rs}^{TM}\rho)\sin(r\phi) + \underline{u}_\phi \frac{r J_r(k_{rs}^{TM}\rho)\cos(r\phi)}{k_{rs}^{TM}\rho} \right\} \quad (6.43)$$

$$\underline{e}_{rs}^{TEe}(\rho, \phi) = \sqrt{\frac{\epsilon_r}{\pi(x_{rs}'^2 - r^2)}} \left(\frac{k_{rs}^{TE}}{J_r(x_{rs}')} \right) \cdot \left\{ \underline{u}_\rho \frac{r J_r(k_{rs}^{TE}\rho)\sin(r\phi)}{k_{rs}^{TE}\rho} + \underline{u}_\phi J_r'(k_{rs}^{TE}\rho)\cos(r\phi) \right\} \quad (6.44)$$

$$\underline{e}_{rs}^{TEo}(\rho, \phi) = -\sqrt{\frac{2}{\pi(x_{rs}'^2 - r^2)}} \left(\frac{k_{rs}^{TE}}{J_r(x_{rs}')} \right)$$

$$\cdot\left\{\underline{u}_\rho \frac{rJ_r(k_{rs}^{TE}\rho)\cos(r\phi)}{k_{rs}^{TE}\rho} + \underline{u}_\phi J_r'(k_{rs}^{TE}\rho)\sin(r\phi)\right\}. \quad (6.45)$$

In view of (6.39) and (3.49), substitution of (6.40) and (6.41) into (6.35) and (6.36) gives

$$\begin{aligned} \underline{E}_{rs}^{TMe-} &= \sqrt{\frac{\epsilon_r}{\pi}}\left(\frac{Z_{rs}^{TMeo}}{aJ_{r+1}(x_{rs})}\right) e^{\gamma_{rs}^{TM}z}\Big\{\underline{u}_\rho J_r'(k_{rs}^{TM}\rho)\cos(r\phi) \\ &\quad -\underline{u}_\phi \frac{rJ_r(k_{rs}^{TM}\rho)\sin(r\phi)}{k_{rs}^{TM}\rho} + \underline{u}_z \frac{k_{rs}^{TM} J_r(k_{rs}^{TM}\rho)\cos(r\phi)}{\gamma_{rs}^{TM}}\Big\} \quad (6.46) \end{aligned}$$

$$\begin{aligned} \underline{E}_{rs}^{TMo-} &= \sqrt{\frac{2}{\pi}}\left(\frac{Z_{rs}^{TMeo}}{aJ_{r+1}(x_{rs})}\right) e^{\gamma_{rs}^{TM}z}\Big\{\underline{u}_\rho J_r'(k_{rs}^{TM}\rho)\sin(r\phi) \\ &\quad +\underline{u}_\phi \frac{rJ_r(k_{rs}^{TM}\rho)\cos(r\phi)}{k_{rs}^{TM}\rho} + \underline{u}_z \frac{k_{rs}^{TM} J_r(k_{rs}^{TM}\rho)\sin(r\phi)}{\gamma_{rs}^{TM}}\Big\}. \quad (6.47) \end{aligned}$$

Substituting (6.44) and (6.45) into (6.37) and (6.38), we obtain

$$\begin{aligned} \underline{E}_{rs}^{TEe-} &= \sqrt{\frac{\epsilon_r}{\pi(x_{rs}'^2 - r^2)}}\left(\frac{k_{rs}^{TE}}{J_r(x_{rs}')}\right) e^{\gamma_{rs}^{TE}z} \\ &\quad \cdot\left\{\underline{u}_\rho \frac{rJ_r(k_{rs}^{TE}\rho)\sin(r\phi)}{k_{rs}^{TE}\rho} + \underline{u}_\phi J_r'(k_{rs}^{TE}\rho)\cos(r\phi)\right\} \quad (6.48) \end{aligned}$$

$$\begin{aligned} \underline{E}_{rs}^{TEo-} &= -\sqrt{\frac{2}{\pi(x_{rs}'^2 - r^2)}}\left(\frac{k_{rs}^{TE}}{J_r(x_{rs}')}\right) e^{\gamma_{rs}^{TE}z} \\ &\quad \cdot\left\{\underline{u}_\rho \frac{rJ_r(k_{rs}^{TE}\rho)\cos(r\phi)}{k_{rs}^{TE}\rho} + \underline{u}_\phi J_r'(k_{rs}^{TE}\rho)\sin(r\phi)\right\}. \quad (6.49) \end{aligned}$$

Expressions for $\underline{E}^{\gamma TM\phi}$, $\underline{E}^{\gamma TE\phi}$, and $\underline{E}^{\gamma TEz}$ in Terms of Waveguide Modes

Equations (6.46) and (6.47) reduce (6.32) to

$$\begin{aligned} \underline{E}^{\gamma TM\phi} &= (-1)^{\gamma r}\sqrt{\frac{\pi}{\epsilon_r}}\left(\frac{z^{TM2}}{Z_{rs}^{TMeo}}\right) aJ_{r+1}(x_{rs}) \\ &\quad \left\{\phi_p^{b2}\underline{E}_{rs}^{TMe-} + (-1)^{\gamma}\phi_p^{b1}\underline{E}_{rs}^{TMo-}\right\} e^{-\gamma_{rs}^{TM}L_3}. \quad (6.50) \end{aligned}$$

Equations (6.48) and (6.49) reduce (6.33) and (6.34) to

$$\underline{E}^{\gamma TE\phi} = (-1)^{\gamma r}\sqrt{\frac{\pi(x_{rs}'^2 - r^2)}{\epsilon_r}}\left(\frac{z^{TE2}J_r(x_{rs}')}{k_{rs}^{TE}}\right) \cdot\left\{(-1)^{\gamma}\phi_p^{b1}\underline{E}_{rs}^{TEe-} - \phi_p^{b2}\underline{E}_{rs}^{TEo-}\right\}e^{-\gamma_{rs}^{TE}L_3} \qquad (6.51)$$

$$\underline{E}^{\gamma TEz} = (-1)^{\gamma r}\sqrt{\frac{\pi(x_{rs}'^2 - r^2)}{\epsilon_r}}\left(\frac{z^{TE4}J_r(x_{rs}')}{k_{rs}^{TE}}\right) \cdot\left\{(-1)^{\gamma}\phi_p^{b4}\underline{E}_{rs}^{TEe-} + \phi_p^{b3}\underline{E}_{rs}^{TEo-}\right\}e^{-\gamma_{rs}^{TE}L_3}. \qquad (6.52)$$

6.1.2 Expression for $\underline{E}^{(3)}(\underline{0}, -\underline{M}^{(1)} - \underline{M}^{(2)})$ in Terms of Waveguide Modes

In this section, expressions (6.50)-(6.52) for $\underline{E}^{\gamma TM\phi}$, $\underline{E}^{\gamma TE\phi}$, and $\underline{E}^{\gamma TEz}$ are substituted into expressions (6.3) and (6.4) for $\underline{E}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma TM})$ and $\underline{E}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma TE})$. The resulting expressions for $\underline{E}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma TM})$ and $\underline{E}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma TE})$ are then substituted into expression (6.2) for $\underline{E}^{(3)}(\underline{0}, -\underline{M}^{(1)} - \underline{M}^{(2)})$.

Substituting (6.50)-(6.52) into (6.3) and (6.4) and using (3.49)-(3.51), we obtain

$$\begin{aligned}\underline{E}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma TM}) = {} & \frac{2}{k_{pq}b}\sqrt{\frac{2\pi b}{c}}\left\{q\sum_{r=0}^{\infty}(-1)^{\gamma r}\sqrt{\frac{\epsilon_r}{2}}\sum_{s=1}^{\infty}\frac{z^{TM2}}{cZ_{rs}^{TMeo}}\right. \\ & \cdot\left(\phi_p^{b2}\underline{E}_{rs}^{TMe} + (-1)^{\gamma}\phi_p^{b1}\underline{E}_{rs}^{TMo-}\right)e^{-\gamma_{rs}^{TM}L_3} - q\sum_{r=1}^{\infty}(-1)^{\gamma r}r\sum_{s=1}^{\infty}\frac{z^{TE2}}{c\sqrt{x_{rs}'^2 - r^2}} \\ & \cdot\left((-1)^{\gamma}\phi_p^{b1}\underline{E}_{rs}^{TEe-} - \phi_p^{b2}\underline{E}_{rs}^{TEo-}\right)e^{-\gamma_{rs}^{TE}L_3} + \frac{pc}{b}\left(\frac{\sin\phi_o}{\phi_o}\right)\sum_{r=0}^{\infty}(-1)^{\gamma r}\sqrt{\frac{\epsilon_r}{2}} \\ & \left.\cdot\sum_{s=1}^{\infty}\frac{x_{rs}'^2 z^{TE4}}{\gamma_{rs}^{TE}ac\sqrt{x_{rs}'^2 - r^2}}\left((-1)^{\gamma}\phi_p^{b4}\underline{E}_{rs}^{TEe-} + \phi_p^{b3}\underline{E}_{rs}^{TEo-}\right)e^{-\gamma_{rs}^{TE}L_3}\right\} \qquad (6.53)\end{aligned}$$

$$\begin{aligned}\underline{E}^{(3)}(\underline{0}, \underline{M}_{pq}^{\gamma TE}) = {} & \frac{2}{k_{pq}b}\sqrt{\frac{2\pi b}{c}}\sqrt{\frac{\epsilon_p\epsilon_q}{4}}\left\{\frac{pc}{b}\sum_{r=0}^{\infty}(-1)^{\gamma r}\sqrt{\frac{\epsilon_r}{2}}\sum_{s=1}^{\infty}\frac{z^{TM2}}{cZ_{rs}^{TMeo}}\right. \\ & \cdot\left(\phi_p^{b2}\underline{E}_{rs}^{TMe} + (-1)^{\gamma}\phi_p^{b1}\underline{E}_{rs}^{TMo-}\right)e^{-\gamma_{rs}^{TM}L_3} - \frac{pc}{b}\sum_{r=1}^{\infty}(-1)^{\gamma r}r\sum_{s=1}^{\infty}\frac{z^{TE2}}{c\sqrt{x_{rs}'^2 - r^2}}\end{aligned}$$

$$\cdot\left((-1)^{\gamma}\phi_p^{b1}\underline{E}_{rs}^{TEe-} - \phi_p^{b2}\underline{E}_{rs}^{TEo-}\right)e^{-\gamma_{rs}^{TE}L_3} - q\left(\frac{\sin\phi_o}{\phi_o}\right)\sum_{r=0}^{\infty}(-1)^{\gamma r}\sqrt{\frac{\epsilon_r}{2}}$$

$$\cdot\sum_{s=1}^{\infty}\frac{x_{rs}'^2 z^{TE4}}{\gamma_{rs}^{TE}ac\sqrt{x_{rs}'^2 - r^2}}\left((-1)^{\gamma}\phi_p^{b4}\underline{E}_{rs}^{TEe-} + \phi_p^{b3}\underline{E}_{rs}^{TEo-}\right)e^{-\gamma_{rs}^{TE}L_3}\Big\}. \quad (6.54)$$

Substitution of (6.53) and (6.54) into (6.2) gives

$$\begin{aligned}\underline{E}^{(3)}(\underline{0}, -\underline{M}^{(1)} - \underline{M}^{(2)}) = {} & 2\sqrt{\frac{2\pi b}{c}}\sum_{r=0}^{\infty}\sqrt{\frac{\epsilon_r}{2}}\sum_{s=1}^{\infty}\Big\{\frac{1}{Z_{rs}^{TMeo}} \\ & \cdot\left(S_{rs}^{TMe}\underline{E}_{rs}^{TMe-} + S_{rs}^{TMo}\underline{E}_{rs}^{TMo-}\right)e^{-\gamma_{rs}^{TM}L_3} \\ & + \left(S_{rs}^{TEe}\underline{E}_{rs}^{TEe-} + S_{rs}^{TEo}\underline{E}_{rs}^{TEo-}\right)e^{-\gamma_{rs}^{TE}L_3}\Big\} \end{aligned} \quad (6.55)$$

where

$$S_{rs}^{TMe} = -\sum_{\gamma=1}^{2}\sum_{q=0}\left(\frac{z^{TM2}}{c}\right)\sum_{\substack{p=0\\p+q\neq 0}}\epsilon_{pq}C_1\phi_p^{b2} \quad (6.56)$$

$$S_{rs}^{TMo} = -\sum_{\gamma=1}^{2}(-1)^{\gamma}\sum_{q=0}\left(\frac{z^{TM2}}{c}\right)\sum_{\substack{p=0\\p+q\neq 0}}\epsilon_{pq}C_1\phi_p^{b1} \quad (6.57)$$

$$\begin{aligned} S_{rs}^{TEe} = {} & \frac{1}{\sqrt{x_{rs}'^2 - r^2}}\sum_{\gamma=1}^{2}(-1)^{\gamma}\sum_{q=0}\sum_{\substack{p=0\\p+q\neq 0}}\epsilon_{pq} \\ & \cdot\left\{r\left(\frac{z^{TE2}}{c}\right)C_1\phi_p^{b1} - \left(\frac{\sin\phi_o}{\phi_o}\right)\left(\frac{z^{TE4}}{c}\right)\frac{x_{rs}'^2 C_2\phi_p^{b4}}{\gamma_{rs}^{TE}a}\right\} \end{aligned} \quad (6.58)$$

$$S_{rs}^{TEo} = \frac{-1}{\sqrt{x_{rs}'^2 - r^2}}\sum_{\gamma=1}^{2}\sum_{q=0}\sum_{\substack{p=0\\p+q\neq 0}}\epsilon_{pq}$$

$$\cdot\left\{r\left(\frac{z^{TE2}}{c}\right)C_1\phi_p^{b2}+\left(\frac{\sin\phi_o}{\phi_o}\right)\left(\frac{z^{TE4}}{c}\right)\frac{x_{rs}'^2 C_2\phi_p^{b3}}{\gamma_{rs}^{TE}a}\right\} \quad (6.59)$$

$$\epsilon_{pq} = \frac{1}{k_{pq}b}\sqrt{\frac{\epsilon_p\epsilon_q}{4}} \quad (6.60)$$

$$C_1 = (-1)^{\gamma r}\left(qV_{pq}^{\gamma TM}+(\frac{pc}{b})V_{pq}^{\gamma TE}\right) \quad (6.61)$$

$$C_2 = (-1)^{\gamma r}\left((\frac{pc}{b})V_{pq}^{\gamma TM}-qV_{pq}^{\gamma TE}\right). \quad (6.62)$$

In (6.61) and (6.62), $V_{pq}^{\gamma TM}=0$ when $p=0$ or $q=0$.

6.2 The Electric Field $\underline{E}^{(3)}(\underline{J}^{\text{imp}},\underline{0})$

Since $x_{01}<ka$, we have, from (3.57),

$$\gamma_{01}^{TM}=j\beta_{01}^{TM}. \quad (6.63)$$

Substitution of (6.63) into [1, eq. (5.10)] gives

$$\begin{aligned}\underline{E}^{(3)}(\underline{J}^{\text{imp}},\underline{0}) &= \frac{2je^{-j\beta_{01}^{TM}L_3}}{\sqrt{\pi}a\omega\epsilon J_1(x_{01})}\{\underline{u}_\rho\beta_{01}^{TM}J_1(k_{01}^{TM}\rho)\sin(\beta_{01}^{TM}(L_3-z))\\ &\quad -\underline{u}_z k_{01}^{TM}J_0(k_{01}^{TM}\rho)\cos(\beta_{01}^{TM}(L_3-z))\}\end{aligned} \quad (6.64)$$

where β_{01}^{TM} is given by (3.59).

6.2.1 Expression for $\underline{E}^{(3)}(\underline{J}^{\text{imp}},\underline{0})$ in Terms of Waveguide Modes

Since, as stated in [1, page 37], the only z-traveling wave contained in $\underline{E}^{(3)}(\underline{J}^{\text{imp}},\underline{0})$ is the unit amplitude z traveling TM_{01} wave, we suspect that $\underline{E}^{(3)}(\underline{J}^{\text{imp}},\underline{0})$ contains only TM_{01} modes. Expression (6.64) for $\underline{E}^{(3)}(\underline{J}^{\text{imp}},\underline{0})$ is recast as

$$\underline{E}^{(3)}(\underline{J}^{\text{imp}},\underline{0}) = \frac{1}{\sqrt{\pi}aJ_1(x_{01})}\left\{\underline{u}_\rho\frac{\beta_{01}^{TM}J_1(k_{01}^{TM}\rho)}{\omega\epsilon}+\underline{u}_z\frac{k_{01}^{TM}J_0(k_{01}^{TM}\rho)}{j\omega\epsilon}\right\}e^{-j\beta_{01}^{TM}z}$$

$$+\frac{e^{-2j\beta_{01}^{TM}L_3}}{\sqrt{\pi}aJ_1(x_{01})}\left\{-\underline{u}_\rho\frac{\beta_{01}^{TM}J_1(k_{01}^{TM}\rho)}{\omega\epsilon}+\underline{u}_z\frac{k_{01}^{TM}J_0(k_{01}^{TM}\rho)}{j\omega\epsilon}\right\}e^{j\beta_{01}^{TM}z}. \quad (6.65)$$

From (6.40) and (6.41), we have, in view of (3.46),

$$\psi_{01}^{TMe}(\rho,\phi) = \frac{J_0(k_{01}^{TM}\rho)}{\sqrt{\pi}x_{01}J_1(x_{01})} \quad (6.66)$$

$$\underline{e}_{01}^{TMe}(\rho,\phi) = -\underline{u}_\rho\frac{J_0'(k_{01}^{TM}\rho)}{\sqrt{\pi}aJ_1(x_{01})} \quad (6.67)$$

so that (6.65) becomes, with the help of (3.51) and (3.49),

$$\begin{aligned}\underline{E}^{(3)}(\underline{J}^{\text{imp}},\underline{0}) = &\left\{\frac{\beta_{01}^{TM}}{\omega\epsilon}\underline{e}_{01}^{TMe}(\rho,\phi)+\underline{u}_z\frac{(k_{01}^{TM})^2\psi_{01}^{TMe}(\rho,\phi)}{j\omega\epsilon}\right\}e^{-j\beta_{01}^{TM}z}\\ +e^{-2j\beta_{01}^{TM}L_3}&\left\{-\frac{\beta_{01}^{TM}}{\omega\epsilon}\underline{e}_{01}^{TMe}(\rho,\phi)+\underline{u}_z\frac{(k_{01}^{TM})^2\psi_{01}^{TMe}(\rho,\phi)}{j\omega\epsilon}\right\}e^{j\beta_{01}^{TM}z}. \quad (6.68)\end{aligned}$$

Substitution of (6.63) into (6.39) gives

$$\frac{\beta_{01}^{TM}}{\omega\epsilon} = Z_{01}^{TMeo}. \quad (6.69)$$

Substituting (6.69) into (6.68) and recalling (6.63), we obtain

$$\underline{E}^{(3)}(\underline{J}^{\text{imp}},\underline{0}) = \underline{E}_{01}^{TMe+} + e^{-2j\beta_{01}^{TM}}\underline{E}_{01}^{TMe-} \quad (6.70)$$

where $\underline{E}_{01}^{TMe+}$ and $\underline{E}_{01}^{TMe-}$ are given by [1, eqs. (B.1) and (B.2)].

6.3 The Electric Field $\underline{E}^{(3)}$

Since (6.47) gives

$$\underline{E}_{01}^{TMo-} = 0 \quad (6.71)$$

and since (3.57) gives (6.63) and

$$\gamma_{11}^{TE} = j\beta_{11}^{TE}, \quad (6.72)$$

substitution of (6.55) and (6.70) into (6.1) yields

$$\begin{aligned}
\underline{E}^{(3)} &= \underline{E}_{01}^{TMe+} + \left(2\sqrt{\frac{\pi b}{c}}\frac{S_{01}^{TMe}e^{-j\beta_{01}^{TM}L_3}}{Z_{01}^{TMeo}} + e^{-2j\beta_{01}^{TM}L_3}\right)\underline{E}_{01}^{TMe-} \\
&+2\sqrt{\frac{2\pi b}{c}}\left(S_{11}^{TEe}\underline{E}_{11}^{TEe-} + S_{11}^{TEo}\underline{E}_{11}^{TEo-}\right)e^{-j\beta_{11}^{TE}L_3} + 2\sqrt{\frac{2\pi b}{c}}\sum_{r=0}^{\infty}\sqrt{\frac{\epsilon_r}{2}} \\
&\cdot\Bigg\{\sum_{\substack{s=1\\(r,s)\neq(0,1)}}\frac{(S_{rs}^{TMe}\underline{E}_{rs}^{TMe-} + S_{rs}^{TMo}\underline{E}_{rs}^{TMo-})e^{-\gamma_{rs}^{TM}L_3}}{Z_{rs}^{TMeo}} \\
&+ \sum_{\substack{s=1\\(r,s)\neq(1,1)}}(S_{rs}^{TEe}\underline{E}_{rs}^{TEe-} + S_{rs}^{TEo}\underline{E}_{rs}^{TEo-})e^{-\gamma_{rs}^{TE}L_3}\Bigg\}
\end{aligned} \tag{6.73}$$

The only z-traveling wave on the right-hand side of (6.73) is the incident wave whose electric field $\underline{E}_{01}^{TMe+}$ is given by [1, eq. (B.1)]

$$\underline{E}_{01}^{TMe+} = \left\{Z_{01}^{TMeo}\underline{e}_{01}^{TMe}(\rho,\phi) + \underline{u}_z\frac{(k_{01}^{TM})^2\phi_{01}^{TMe}(\rho,\phi)}{j\omega\epsilon}\right\}e^{-j\beta_{01}^{TM}z} \tag{6.74}$$

The time-average z-directed power associated with $\underline{E}_{01}^{TMe+}$ is, as given by a formula similar to (5.31), Z_{01}^{TMeo}. Moreover, $\underline{E}_{01}^{TMe+}$ has the phase factor $e^{-j\beta_{01}^{TM}L_3}$ when $z = L_3$. However, the normalized electric field $\underline{E}_{01}^{TMe+}e^{j\beta_{01}^{TM}L_3}/\sqrt{Z_{01}^{TMeo}}$ has unit power and no phase factor when $z = L_3$. Thus, the z-traveling wave part of $\underline{E}^{(3)}e^{j\beta_{01}^{TM}L_3}/\sqrt{Z_{01}^{TMeo}}$ has unit power and no phase factor when $z = L_3$. Multiplying both sides of (6.73) by $e^{j\beta_{01}^{TM}L_3}/\sqrt{Z_{01}^{TMeo}}$, we obtain the normalized electric field $\underline{E}^{(3)}e^{j\beta_{01}^{TM}L_3}/\sqrt{Z_{01}^{TMeo}}$ given by

$$\begin{aligned}
\left(\frac{e^{j\beta_{01}^{TM}L_3}}{\sqrt{Z_{01}^{TMeo}}}\right)\underline{E}^{(3)} &= \left(\frac{e^{j\beta_{01}^{TM}L_3}}{\sqrt{Z_{01}^{TMeo}}}\right)\underline{E}_{01}^{TMe+} - \left(\frac{C_{01}^{TMe}e^{-j\beta_{01}^{TM}L_3}}{\sqrt{Z_{01}^{TMeo}}}\right)\underline{E}_{01}^{TMe-} \\
&+ \left(\frac{e^{-j\beta_{11}^{TE}L_3}}{\sqrt{Y_{11}^{TEeo}}}\right)\left(C_{11}^{TEe}\underline{E}_{11}^{TEe-} + C_{11}^{TEo}\underline{E}_{11}^{TEo-}\right) \\
-\sum_{r=0}^{\infty}\sum_{\substack{s=1\\(r,s)\neq(0,1)}}&\left\{\left(\frac{e^{-\gamma_{rs}^{TM}L_3}}{\sqrt{|Z_{rs}^{TMeo}|}}\right)\left(C_{rs}^{TMe}\underline{E}_{rs}^{TMe-} + C_{rs}^{TMo}\underline{E}_{rs}^{TMo-}\right)\right\}
\end{aligned}$$

$$+\sum_{r=0}^{\infty} \sum_{\substack{s=1 \\ (r,s)\neq(1,1)}}^{\infty} \left\{ \left(\frac{e^{-\gamma_{rs}^{TE} L_3}}{\sqrt{|Y_{rs}^{TEeo}|}} \right) \left(C_{rs}^{TEe} \underline{E}_{rs}^{TEe-} + C_{rs}^{TEo} \underline{E}_{rs}^{TEo-} \right) \right\} \quad (6.75)$$

where Z_{rs}^{TMeo} is given by (6.39) and [1, eq. (B.54)]

$$Y_{rs}^{TEeo} = \frac{\gamma_{rs}^{TE}}{j\omega\mu}. \quad (6.76)$$

In (6.39), γ_{01}^{TM} is to be replaced by $j\beta_{01}^{TM}$. In (6.76), γ_{11}^{TE} is to be replaced by $j\beta_{11}^{TE}$. The C's in (6.75) are given by

$$C_{01}^{TMe} = -1 - 2\sqrt{\frac{\pi b}{c}} \left(\frac{S_{01}^{TMe} e^{j\beta_{01}^{TM} L_3}}{Z_{01}^{TMeo}} \right) \quad (6.77)$$

$$C_{11}^{TEe} = 2\sqrt{\frac{2\pi b}{c}} \sqrt{\frac{Y_{11}^{TEeo}}{Z_{01}^{TMeo}}} S_{11}^{TEe} e^{j\beta_{01}^{TM} L_3} \quad (6.78)$$

$$C_{11}^{TEo} = 2\sqrt{\frac{2\pi b}{c}} \sqrt{\frac{Y_{11}^{TEeo}}{Z_{01}^{TMeo}}} S_{11}^{TEo} e^{j\beta_{01}^{TM} L_3} \quad (6.79)$$

$$C_{rs}^{TMe} = -2\sqrt{\frac{2\pi b}{c}} \sqrt{\frac{\epsilon_r}{2}} \sqrt{\frac{|Z_{rs}^{TMeo}|}{Z_{01}^{TMeo}}} \left(\frac{S_{rs}^{TMe}}{Z_{rs}^{TMeo}} \right) e^{j\beta_{01}^{TM} L_3} \quad (6.80)$$

$$C_{rs}^{TMo} = -2\sqrt{\frac{2\pi b}{c}} \sqrt{\frac{\epsilon_r}{2}} \sqrt{\frac{|Z_{rs}^{TMeo}|}{Z_{01}^{TMeo}}} \left(\frac{S_{rs}^{TMo}}{Z_{rs}^{TMeo}} \right) e^{j\beta_{01}^{TM} L_3} \quad (6.81)$$

$$C_{rs}^{TEe} = 2\sqrt{\frac{2\pi b}{c}} \sqrt{\frac{\epsilon_r}{2}} \sqrt{\frac{|Y_{rs}^{TEeo}|}{Z_{01}^{TMeo}}} S_{rs}^{TEe} e^{j\beta_{01}^{TM} L_3} \quad (6.82)$$

$$C_{rs}^{TEo} = 2\sqrt{\frac{2\pi b}{c}} \sqrt{\frac{\epsilon_r}{2}} \sqrt{\frac{|Y_{rs}^{TEeo}|}{Z_{01}^{TMeo}}} S_{rs}^{TEo} e^{j\beta_{01}^{TM} L_3} \quad (6.83)$$

The magnitudes of the squares of the constants C_{01}^{TMe}, C_{11}^{TEe}, and C_{11}^{TEo} are time-average powers (see Section 6.3.1). The minus sign which precedes the $\underline{E}_{01}^{TMe-}$ term on the right-hand side of (6.75) makes the phase of C_{01}^{TMe} equal to the phase of the coefficient of $\underline{e}_{01}^{TMe}(\rho, \phi)$. See (6.35). The $\underline{E}_{rs}^{TMe-}$, $\underline{E}_{rs}^{TEe-}$, and $\underline{E}_{rs}^{TEo-}$ terms in (6.75) were patterned after the $\underline{E}_{01}^{TMe-}$, $\underline{E}_{11}^{TEe-}$, and $\underline{E}_{11}^{TEo-}$ terms, respectively.

6.3.1 Time-Average Power

The time-average power of the normalized electric field $\underline{E}^{(3)}e^{j\beta_{01}^{TM}L_3}/\sqrt{Z_{01}^{TMeo}}$ of (6.75) is given by a formula similar to (5.31). This power consists only of the time-average powers associated with the $\underline{E}_{01}^{TMe+}$, $\underline{E}_{01}^{TMe-}$, $\underline{E}_{11}^{TEe-}$, and $\underline{E}_{11}^{TEo-}$ terms on the right-hand side of (6.75). The time-average z-directed power associated with the $\underline{E}_{01}^{TMe+}$ term in (6.75) is 1 W. The time-average $-z$-directed powers associated with the $\underline{E}_{01}^{TMe-}$, $\underline{E}_{11}^{TEe-}$, and $\underline{E}_{11}^{TEo-}$ terms in (6.75) are $|C_{01}^{TMe}|^2$, $|C_{11}^{TEe}|^2$, and $|C_{11}^{TEo}|^2$, respectively.

6.3.2 The Coefficient C_{01}^{TMe}

In this subsection, C_{01}^{TMe} of (6.77) is expressed in a form suitable for calculation. In (6.77), S_{01}^{TMe} is given by (6.56) in which the quantities z^{TM2}/c and ϕ_p^{b2} appear. Comparing (6.15) with (3.76), we see that, since $x_{01} < ka$,

$$\frac{z^{\delta 2}}{c} = \frac{1}{2}\left[\hat{G}_q^{\delta}\right]_{01} \tag{6.84}$$

where $\left[\hat{G}_q^{\delta}\right]_{01}$ is $\hat{G}_q^{\delta}$ when $r = 0$ and $s = 1$. Here, δ is either TM or TE. Setting $r = 0$ in (6.29), we obtain

$$\phi_p^{b2} = \phi_p^{(2)}. \tag{6.85}$$

Substitution of (6.61), (6.84), and (6.85) into (6.56) gives

$$S_{01}^{TMe} = \sum_{\gamma=1}\sum_{q=0}\left[\hat{G}_q^{TM}\right]_{01}\sum_{\substack{p=0\\ p+q\neq 0}}\epsilon_{pq}\phi_p^{(2)}\left(qV_{pq}^{\gamma TM} + (\frac{pc}{b})V_{pq}^{\gamma TE}\right). \tag{6.86}$$

The quantity Z_{01}^{TMeo} in (6.77) is given by (6.69):

$$Z_{01}^{TMeo} = \frac{\eta\beta_{01}^{TM}}{k} \tag{6.87}$$

Substituting (6.86) and (6.87) into (6.77), we obtain

$$C_{01}^{TMe} = -1 + \sum_{\gamma=1}^{2} \sum_{q=0} \sum_{\substack{p=0 \\ p+q\neq 0}} \left\{ C_{01,pq}^{TMe,\gamma TM} \left(\frac{V_{pq}^{\gamma TM} e^{j\beta_{01}^{TM} L_3}}{\eta} \right) \right.$$

$$\left. + C_{01,pq}^{TMe,\gamma TE} \left(\frac{V_{pq}^{\gamma TE} e^{j\beta_{01}^{TM} L_3}}{\eta} \right) \right\} \tag{6.88}$$

where

$$C_{01,pq}^{TMe,\gamma TM} = \sqrt{\frac{\pi b}{c}} \left(\frac{k}{\beta_{01}^{TM}} \right) q \epsilon_{pq} \left[\hat{G}_q^{TM} \right]_{01} \phi_p^{(2)} \tag{6.89}$$

$$C_{01,pq}^{TMe,\gamma TE} = \sqrt{\frac{\pi b}{c}} \left(\frac{k}{\beta_{01}^{TM}} \right) \left(\frac{pc}{b} \right) \epsilon_{pq} \left[\hat{G}_q^{TM} \right]_{01} \phi_p^{(2)}. \tag{6.90}$$

6.3.3 The Coefficients C_{11}^{TEe} and C_{11}^{TEo}

In this section, C_{11}^{TEe} of (6.78) and C_{11}^{TEo} of (6.79) are expressed in forms suitable for calculation. In (6.78) and (6.79), S_{11}^{TEe} and S_{11}^{TEo} are given by (6.58) and (6.59), respectively. The quantities z^{TE4}/c and $\gamma_{11}^{TE}a$ appear in both (6.58) and (6.59). Comparing (6.16) with (3.108), we see that, since $x'_{11} < ka$,

$$\frac{z^{TE4}}{c} = -\frac{j}{2} \left[\hat{G}_q^{(4)} \right]_{11} \tag{6.91}$$

where $\left[\hat{G}_q^{(4)} \right]_{11}$ is $\hat{G}_q^{(4)}$ when $r = 1$ and $s = 1$. From (3.58), we have

$$\gamma_{11}^{TE} = j\beta_{11}^{TE}. \tag{6.92}$$

Substitution of (6.61), (6.62), (6.84), (6.91), and (6.92) into (6.58) and (6.59) gives

$$S_{11}^{TEe} = \frac{1}{2\sqrt{x_{11}'^2 - 1}} \sum_{\gamma=1}^{2} \sum_{q=0} \sum_{\substack{p=0 \\ p+q\neq 0}} \epsilon_{pq} \left\{ \left[\hat{G}_q^{TE} \right]_{11} \phi_p^{b1} \left(q V_{pq}^{\gamma TM} + \left(\frac{pc}{b} \right) V_{pq}^{\gamma TE} \right) \right.$$

$$+\left(\frac{\sin\phi_0}{\phi_0}\right)\frac{\left[\hat{G}_q^{(4)}\right]_{11} x_{11}'^2 \phi_p^{b4}\left(\left(\frac{pc}{b}\right)V_{pq}^{\gamma TM} - qV_{pq}^{\gamma TE}\right)}{\beta_{11}^{TE} a}\Bigg\} \tag{6.93}$$

$$S_{11}^{TEo} = \frac{-1}{2\sqrt{x_{11}'^2-1}}\sum_{\gamma=1}^{2}\sum_{q=0}\sum_{\substack{p=0\\p+q\neq 0}} \epsilon_{pq}\Bigg\{\left[\hat{G}_q^{TE}\right]_{11}\phi_p^{b2}\left(qV_{pq}^{\gamma TM} + \left(\frac{pc}{b}\right)V_{pq}^{\gamma TE}\right)$$

$$-\left(\frac{\sin\phi_0}{\phi_0}\right)\frac{\left[\hat{G}_q^{(4)}\right]_{11} x_{11}'^2 \phi_p^{b3}\left(\left(\frac{pc}{b}\right)V_{pq}^{\gamma TM} - qV_{pq}^{\gamma TE}\right)}{\beta_{11}^{TE} a}\Bigg\}. \tag{6.94}$$

The admittance Y_{11}^{TEeo} in (6.78) and (6.79) is obtained by substituting (6.92) into [1, eq. (B.54)]:

$$Y_{11}^{TEeo} = \frac{\beta_{11}^{TE}}{k\eta}. \tag{6.95}$$

Substituting (6.87), (6.95), and (6.93) into (6.78), we obtain

$$C_{11}^{TEe} = \sum_{\gamma=1}^{2}\sum_{q=0}\sum_{\substack{p=0\\p+q\neq 0}}\Bigg\{C_{11,pq}^{TEe,\gamma TM}\left(\frac{V_{pq}^{\gamma TM} e^{j\beta_{01}^{TM}L_3}}{\eta}\right)$$

$$+C_{11,pq}^{TEe,\gamma TE}\left(\frac{V_{pq}^{\gamma TE} e^{j\beta_{01}^{TM}L_3}}{\eta}\right)\Bigg\} \tag{6.96}$$

where

$$C_{11}^{TEe,\gamma TM} = \sqrt{\frac{2\pi b}{c}}\sqrt{\frac{\beta_{11}^{TE}}{\beta_{01}^{TM}}}\frac{\epsilon_{pq}}{\sqrt{x_{11}'^2-1}}\Bigg\{q(\hat{G}_q^{TE})_{11}\phi_p^{b1}$$

$$+(\frac{pc}{b})\left(\frac{\sin\phi_o}{\phi_o}\right)\frac{x_{11}'^2(\hat{G}_q^{(4)})_{11}\phi_p^{b4}}{\beta_{11}^{TE} a}\Bigg\} \tag{6.97}$$

$$C_{11}^{TEe,\gamma TE} = \sqrt{\frac{2\pi b}{c}}\sqrt{\frac{\beta_{11}^{TE}}{\beta_{01}^{TM}}}\frac{\epsilon_{pq}}{\sqrt{x_{11}'^2-1}}\Bigg\{(\frac{pc}{b})(\hat{G}_q^{TE})_{11}\phi_p^{b1}$$

$$-q\left(\frac{\sin\phi_o}{\phi_o}\right)\frac{x_{11}'^2(\hat{G}_q^{(4)})_{11}\phi_p^{b4}}{\beta_{11}^{TE} a}\Bigg\}. \tag{6.98}$$

Substituting (6.87), (6.95), and (6.94) into (6.79), we obtain

$$C_{11}^{TEo} = \sum_{\gamma=1}^{2}\sum_{q=0}\sum_{\substack{p=0\\p+q\neq 0}} \left\{ C_{11,pq}^{TEo,\gamma TM}\left(\frac{V_{pq}^{\gamma TM} e^{j\beta_{01}^{TM}L_3}}{\eta}\right) \right.$$

$$\left. + C_{11,pq}^{TEo,\gamma TE}\left(\frac{V_{pq}^{\gamma TE} e^{j\beta_{01}^{TM}L_3}}{\eta}\right) \right\} \tag{6.99}$$

where

$$C_{11}^{TEo,\gamma TM} = -\sqrt{\frac{2\pi b}{c}}\sqrt{\frac{\beta_{11}^{TE}}{\beta_{01}^{TM}}}\frac{(-1)^{\gamma}\epsilon_{pq}}{\sqrt{x_{11}'^{2}-1}}\left\{ q(\hat{G}_q^{TE})_{11}\phi_p^{b2} \right.$$

$$\left. - \left(\frac{pc}{b}\right)\left(\frac{\sin\phi_o}{\phi_o}\right)\frac{x_{11}'^{2}(\hat{G}_q^{(4)})_{11}\phi_p^{b3}}{\beta_{11}^{TE}a}\right\} \tag{6.100}$$

$$C_{11}^{TEe,\gamma TE} = -\sqrt{\frac{2\pi b}{c}}\sqrt{\frac{\beta_{11}^{TE}}{\beta_{01}^{TM}}}\frac{(-1)^{\gamma}\epsilon_{pq}}{\sqrt{x_{11}'^{2}-1}}\left\{ \left(\frac{pc}{b}\right)(\hat{G}_q^{TE})_{11}\phi_p^{b2} \right.$$

$$\left. + q\left(\frac{\sin\phi_o}{\phi_o}\right)\frac{x_{11}'^{2}(\hat{G}_q^{(4)})_{11}\phi_p^{b3}}{\beta_{11}^{TE}a}\right\}. \tag{6.101}$$

The ϕ_p's in (6.97), (6.98), (6.100), and (6.101) are obtained by setting $r = 1$ in (6.28)-(6.31). From (2.8) and (2.9), we have

$$\frac{b}{2x_o} = \phi_o \tag{6.102}$$

so that

$$\sin\left(\frac{b}{2x_o}\right) = \frac{b}{2a}. \tag{6.103}$$

Assuming that $\phi_o \leq \pi/2$, we have

$$\cos\left(\frac{b}{2x_o}\right) = \sqrt{1-\left(\frac{b}{2a}\right)^2}. \tag{6.104}$$

Substitution of (6.103) and (6.104) into (6.28)–(6.31) yields

$$\phi_o^{b1} = \sqrt{1 - \left(\frac{b}{2a}\right)^2}\,\phi_p^{(1)} - \left(\frac{b}{2a}\right)\phi_p^{(2)} \tag{6.105}$$

$$\phi_o^{b2} = \sqrt{1 - \left(\frac{b}{2a}\right)^2}\,\phi_p^{(2)} + \left(\frac{b}{2a}\right)\phi_p^{(1)} \tag{6.106}$$

$$\phi_o^{b3} = \sqrt{1 - \left(\frac{b}{2a}\right)^2}\,\phi_p^{(3)} - \left(\frac{b}{2a}\right)\phi_p^{(4)} \tag{6.107}$$

$$\phi_o^{b4} = \sqrt{1 - \left(\frac{b}{2a}\right)^2}\,\phi_p^{(3)} + \left(\frac{b}{2a}\right)\phi_p^{(3)}. \tag{6.108}$$

Chapter 7

The Tangential Electric Field in the Apertures

The tangential part of the electric field in the left-hand aperture A_1 is called $\underline{E}_t^{(A1)}(\phi, z)$. The tangential part of the electric field in the right-hand aperture A_2 is called $\underline{E}_t^{(A2)}(\phi, z)$. These tangential parts are given by

$$\underline{E}_t^{(A\gamma)}(\phi, z) = \underline{u}_\rho \times \underline{M}^{(\gamma)}, \quad \gamma = 1, 2 \tag{7.1}$$

where $\underline{u}_\rho$ is the unit vector in the ρ-direction and $\underline{M}^{(\gamma)}$ is given by [1, eqs. (2.11) and (2.12)]. Substituting [1, eqs. (2.11) and (2.12)] into (7.1), we obtain

$$\begin{aligned} \underline{E}_t^{(A\gamma)}(\phi, z) = \sum_{p=0} \sum_{\substack{q=0 \\ p+q\neq 0}} \Big\{ & V_{pq}^{\gamma TM} \left(\underline{u}_\rho \times \underline{M}_{pq}^{\gamma TM}(\phi, z) \right) \\ & + V_{pq}^{\gamma TE} \left(\underline{u}_\rho \times \underline{M}_{pq}^{\gamma TE}(\phi, z) \right) \Big\} \end{aligned} \tag{7.2}$$

where the double summation is truncated as in (5.1). In (7.2),

$$V_{pq}^{\gamma TM} = 0, \quad p = 0 \text{ or } q = 0. \tag{7.3}$$

Substitution of [1, eqs. (2.13) and (2.14)] into (7.2) gives

$$\underline{E}_t^{(A\gamma)}(\phi, z) = \underline{u}_\phi (-1)^\gamma \left(\frac{\sin \phi_o}{\phi_o} \right)$$

$$\cdot \sum_{p=0} \sum_{\substack{q=0 \\ p+q\neq 0}} \left\{ V_{pq}^{\gamma TM} e_{ypq}^{TM}(y^{\gamma+}, z^+) + V_{pq}^{\gamma TE} e_{ypq}^{TE}(y^{\gamma+}, z^+) \right\}$$

$$+\underline{u}_z \sum_{p=0} \sum_{\substack{q=0 \\ p+q\neq 0}} \left\{ V_{pq}^{\gamma TM} e_{zpq}^{TM}(y^{\gamma+}, z^+) + V_{pq}^{\gamma TE} e_{zpq}^{TE}(y^{\gamma+}, z^+) \right\} \quad (7.4)$$

where $y^{\gamma+}$ and z^+ are given by [1, eqs. (2.15)-(2.17)]

$$y^{\gamma+} = (2-\gamma)\pi x_o + (-1)^\gamma \phi x_o + \frac{b}{2} \quad (7.5)$$

$$z^+ = z + \frac{c}{2} \quad (7.6)$$

where, from (6.102),

$$x_o = \frac{b}{2\phi_o}. \quad (7.7)$$

Now, e_{ypq}^{TM} and e_{zpq}^{TM} are the y- and z-components of $\underline{e}_{pq}^{TM}$ given by (A.10), and e_{ypq}^{TE} and e_{zpq}^{TE} are the y- and z-components of $\underline{e}_{pq}^{TE}$ given by (A.23) so that (7.4) becomes

$$\underline{E}_t^{(A\gamma)}(\phi, z) = \underline{u}_\phi (-1)^\gamma 2\pi \sqrt{\frac{b}{c}} \left(\frac{\sin\phi_o}{\phi_o} \right)$$

$$\cdot \sum_{p=0} \sum_{\substack{q=0 \\ p+q\neq 0}} \sqrt{\frac{\epsilon_p \epsilon_q}{4}} \left(\frac{1}{k_{pq} b} \right) \left(-\frac{p}{b} V_{pq}^{\gamma TM} + \frac{q}{c} V_{pq}^{\gamma TE} \right) \cos\frac{p\pi y^{\gamma+}}{b} \sin\frac{q\pi z^+}{c}$$

$$-\underline{u}_z 2\pi \sqrt{\frac{b}{c}} \sum_{p=0} \sum_{\substack{q=0 \\ p+q\neq 0}} \sqrt{\frac{\epsilon_p \epsilon_q}{4}} \left(\frac{1}{k_{pq} b} \right) \left(\frac{q}{c} V_{pq}^{\gamma TM} + \frac{p}{b} V_{pq}^{\gamma TE} \right) \sin\frac{p\pi y^{\gamma+}}{b} \cos\frac{q\pi z^+}{c}.$$

$$(7.8)$$

We want to normalize $\underline{E}_t^{(A\gamma)}(\phi, z)$ of (7.8) by dividing by $-|\underline{E}_{01}^{TMe+}|_{\text{rms}}$ where the subscript "rms" denotes the root mean square value of the transverse part over the waveguide cross section at $z = 0$. Recall that $\underline{E}_{01}^{TMe+}$ is the electric field of the z traveling wave in the circular waveguide. We choose to divide by $-|\underline{E}_{01}^{TMe+}|_{\text{rms}}$ rather than $|\underline{E}_{01}^{TMe+}|_{\text{rms}}$ because the z-directed electric current associated with $\underline{E}_{01}^{TMe+}$ is negative at $(\rho, z) = (a, 0)$. This electric current is given by $-\underline{u}_\rho \times \underline{H}_{01}^{TMe+}$ where $\underline{H}_{01}^{TMe+}$ is given by [1, eqs. (5.2) and (5.9)].

According to (6.74), the transverse part of $\underline{E}_{01}^{TMe+}$ is $Z_{01}^{TMeo}\underline{e}_{01}^{TMe}(\rho,\phi)$ $\cdot e^{-j\beta_{01}^{TM}z}$ so that

$$|\underline{E}_{01}^{TMe+}|_{\rm rms} = |Z_{01}^{TMeo}|\left\{\frac{1}{\pi a^2}\int_0^a \rho\, d\rho \int_0^{2\pi} d\phi \left(\underline{e}_{01}^{TMe}\cdot\underline{e}_{01}^{TMe}\right)\right\}^{1/2}. \tag{7.9}$$

One of the orthogonality relationships in [1, eq. (B.64)] and

$$Z_{01}^{TMeo} = \frac{\eta\beta_{01}^{TM}}{k} \tag{7.10}$$

reduce (7.9) to

$$|\underline{E}_{01}^{TMe+}|_{\rm rms} = \frac{\eta\beta_{01}^{TM}}{\sqrt{\pi}ka}. \tag{7.11}$$

Equation (7.10) was obtained by replacing $1/(\omega\epsilon)$ by η/k in (6.69). Substitution of (7.11) and (7.5)–(7.7) into (7.8) gives

$$\begin{aligned}
\frac{\underline{E}_t^{(A\gamma)}(\phi,z)}{|\underline{E}_{01}^{TMe+}|_{\rm rms}} = {} & \underline{u}_\phi(-1)^\gamma 2\pi\sqrt{\frac{\pi b}{c}}\left(\frac{k}{\beta_{01}^{TM}}\right)\left(\frac{\sin\phi_o}{\phi_o}\right)e^{-j\beta_{01}^{TM}L_3}\sum_{p=0}\sum_{\substack{q=0\\p+q\neq 0}}\sqrt{\frac{\epsilon_p\epsilon_q}{4}} \\
& \cdot\left(\frac{1}{k_{pq}b}\right)\left\{\left(\frac{pa}{b}\right)\left(\frac{V_{pq}^{\gamma TM}e^{j\beta_{01}^{TM}L_3}}{\eta}\right)-\left(\frac{qa}{c}\right)\left(\frac{V_{pq}^{\gamma TE}e^{j\beta_{01}^{TE}L_3}}{\eta}\right)\right\} \\
& \cdot\cos\left\{\left(\frac{p\pi}{2\phi_o}\right)\left((-1)^\gamma\phi+(2-\gamma)\pi\right)+\frac{p\pi}{2}\right\}\sin\left(\frac{q\pi z}{c}+\frac{q\pi}{2}\right) \\
& +\underline{u}_z 2\pi\sqrt{\frac{\pi b}{c}}\left(\frac{k}{\beta_{01}^{TM}}\right)e^{-j\beta_{01}^{TM}L_3}\sum_{p=0}\sum_{\substack{q=0\\p+q\neq 0}}\sqrt{\frac{\epsilon_p\epsilon_q}{4}} \\
& \cdot\left(\frac{1}{k_{pq}b}\right)\left\{\left(\frac{qa}{c}\right)\left(\frac{V_{pq}^{\gamma TM}e^{j\beta_{01}^{TM}L_3}}{\eta}\right)+\left(\frac{pa}{b}\right)\left(\frac{V_{pq}^{\gamma TE}e^{j\beta_{01}^{TE}L_3}}{\eta}\right)\right\} \\
& \cdot\sin\left\{\left(\frac{p\pi}{2\phi_o}\right)\left((-1)^\gamma\phi+(2-\gamma)\pi\right)+\frac{p\pi}{2}\right\}\cos\left(\frac{q\pi z}{c}+\frac{q\pi}{2}\right).
\end{aligned} \tag{7.12}$$

When $z=0$, the ϕ-component of (7.12) is $E_\phi^{(A\gamma)}(\phi,0)/|\underline{E}_{01}^{TEe+}|_{\rm rms}$ given by

$$\frac{E_\phi^{(A\gamma)}(\phi,0)}{|\underline{E}_{01}^{TMe+}|_{\text{rms}}} = (-1)^\gamma 2\pi\sqrt{\frac{\pi b}{c}}\left(\frac{k}{\beta_{01}^{TM}}\right)\left(\frac{\sin\phi_o}{\phi_o}\right)e^{-j\beta_{01}^{TM}L_3}\sum_{p=0}\sum_{\substack{q=0\\p+q\neq 0}}\sqrt{\frac{\epsilon_p\epsilon_q}{4}}$$
$$\cdot\left(\frac{1}{k_{pq}b}\right)\left\{\left(\frac{pa}{b}\right)\left(\frac{V_{pq}^{\gamma TM}e^{j\beta_{01}^{TM}L_3}}{\eta}\right)-\left(\frac{qa}{c}\right)\left(\frac{V_{pq}^{\gamma TE}e^{j\beta_{01}^{TE}L_3}}{\eta}\right)\right\}$$
$$\cdot\cos\left\{\left(\frac{p\pi}{2}\right)\left(\frac{(-1)^\gamma\phi+(2-\gamma)\pi}{\phi_o}\right)+\frac{p\pi}{2}\right\}\sin\frac{q\pi}{2}. \tag{7.13}$$

When $\phi=(2-\gamma)\pi$, the z-component of (7.12) is $E_z^{(A\gamma)}((2-\gamma)\pi,z)/|\underline{E}_{01}^{TMe+}|_{\text{rms}}$ given by

$$\frac{E_z^{(A\gamma)}((2-\gamma)\pi,z)}{|\underline{E}_{01}^{TMe+}|_{\text{rms}}} = 2\pi\sqrt{\frac{\pi b}{c}}\left(\frac{k}{\beta_{01}^{TM}}\right)e^{-j\beta_{01}^{TM}L_3}\sum_{p=0}\sum_{\substack{q=0\\p+q\neq 0}}\sqrt{\frac{\epsilon_p\epsilon_q}{4}}$$
$$\cdot\left(\frac{1}{k_{pq}b}\right)\left\{\left(\frac{qa}{c}\right)\left(\frac{V_{pq}^{\gamma TM}e^{j\beta_{01}^{TM}L_3}}{\eta}\right)+\left(\frac{pa}{b}\right)\left(\frac{V_{pq}^{\gamma TE}e^{j\beta_{01}^{TE}L_3}}{\eta}\right)\right\}$$
$$\cdot\sin\frac{p\pi}{2}\cos\left(\frac{q\pi z}{c}+\frac{q\pi}{2}\right). \tag{7.14}$$

Chapter 8

Numerical Results

A computer program was written to calculate the time-average power transmitted into the rectangular waveguides, the time-average power reflected in the circular waveguide, and the ϕ and z components of the electric field in the apertures. This computer program will be described and listed in a subsequent report. Some numerical results obtained by means of this computer program are presented in this chapter.

When the time-average incident power in the circular waveguide is unity, the time-average power transmitted into the rectangular waveguides is P_t given by (5.49) and the time-average power reflected in the circular waveguide is called P_r. According to the discussion in Section 6.3.1,

$$P_r = |C_{01}^{TMe}|^2 + |C_{11}^{TEe}|^2 + |C_{11}^{TEo}|^2. \tag{8.1}$$

Figures 8.1 to 8.5 show plots of P_t and P_r versus ka for various values of L_3 when

$$\left.\begin{aligned} \frac{b}{a} &= 1.1 \\ \frac{c}{a} &= 0.5 \end{aligned}\right\} \tag{8.2}$$

The plots of Figs. 8.1 to 8.5 are for $L_3/[\lambda_{01}^{TM}]_{ka=2.95} = 0.35, 0.40, 0.45, 0.50,$ and 0.55, respectively. Here, $[\lambda_{01}^{TM}]_{ka=2.95}$ is the wavelength of the TM_{01} mode in the circular waveguide when $ka = 2.95$:

$$[\lambda_{01}^{TM}]_{ka=2.95} = \frac{2\pi a}{\sqrt{(2.95)^2 - x_{01}^2}}. \tag{8.3}$$

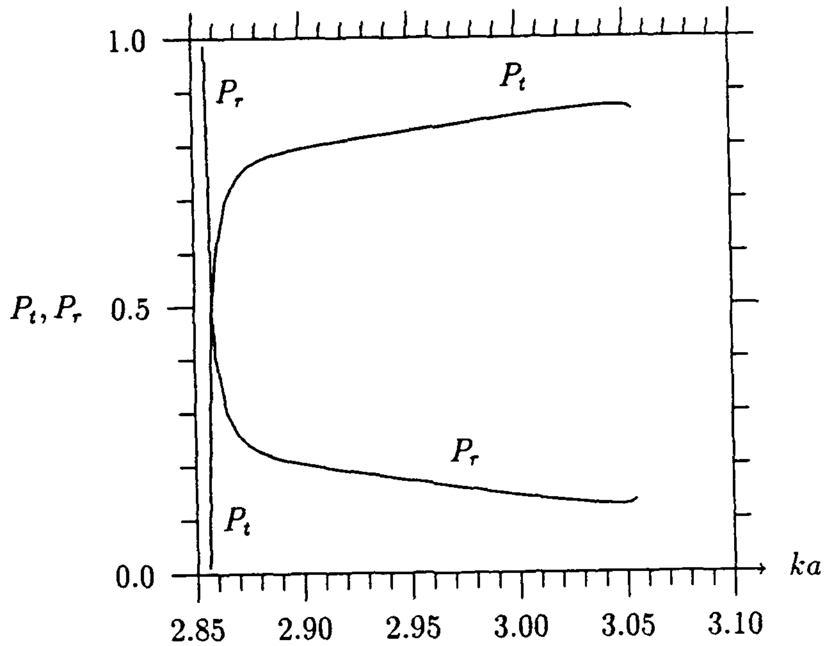


Fig. 8.1. Plots of the ratio P_t of the transmitted power to the input power and the ratio P_r of the reflected power to the input power when $L_3 = 0.35 \left[\lambda_{01}^{TM}\right]_{ka=2.95}$. The input power is the power of the incident TM_{01} wave in the circular waveguide. $P_t + P_r = 1$.

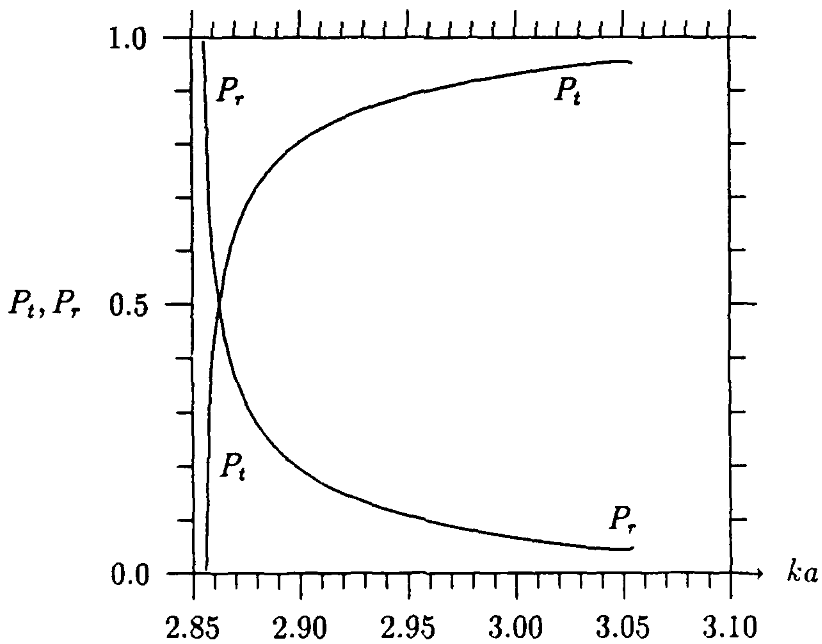


Fig. 8.2. Plots of the ratio P_t of the transmitted power to the input power and the ratio P_r of the reflected power to the input power when $L_3 = 0.40 \left[\lambda_{01}^{TM}\right]_{ka=2.95}$. The input power is the power of the incident TM_{01} wave in the circular waveguide. $P_t + P_r = 1$.

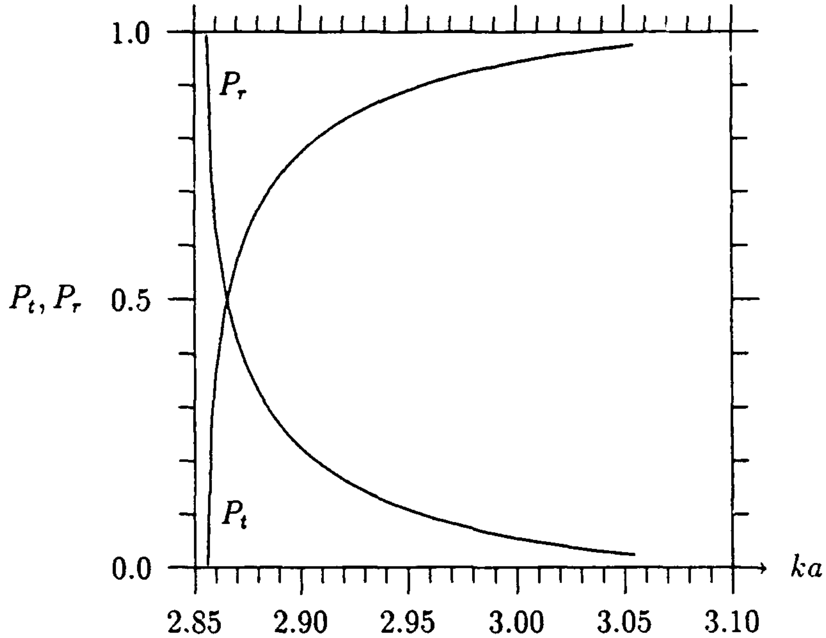


Fig. 8.3. Plots of the ratio P_t of the transmitted power to the input power and the ratio P_r of the reflected power to the input power when $L_3 = 0.45 \left[\lambda_{01}^{TM}\right]_{ka=2.95}$. The input power is the power of the incident TM_{01} wave in the circular waveguide. $P_t + P_r = 1$.

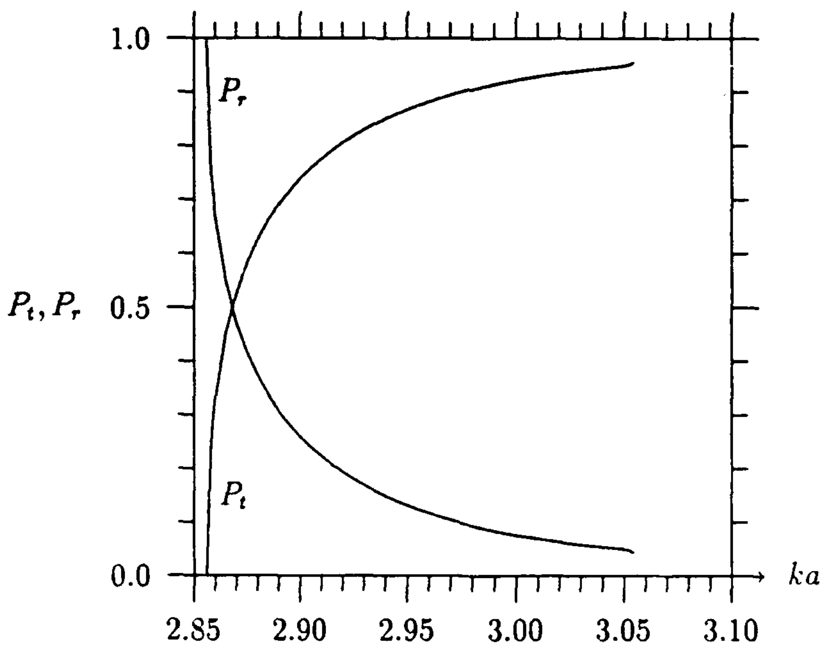


Fig. 8.4. Plots of the ratio P_t of the transmitted power to the input power and the ratio P_r of the reflected power to the input power when $L_3 = 0.50 \left[\lambda_{01}^{TM}\right]_{ka=2.95}$. The input power is the power of the incident TM_{01} wave in the circular waveguide. $P_t + P_r = 1$.

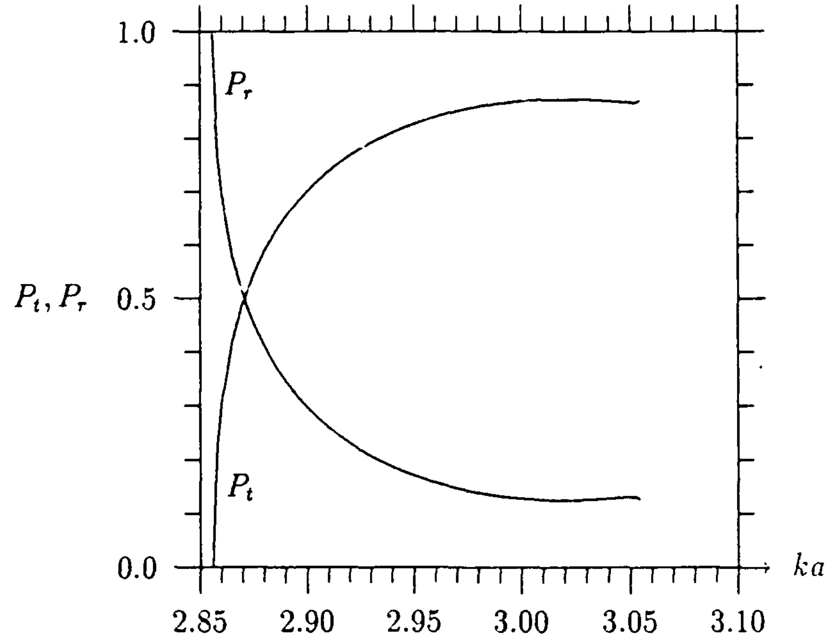


Fig. 8.5. Plots of the ratio P_t of the transmitted power to the input power and the ratio P_r of the reflected power to the input power when $L_3 = 0.55 \left[\lambda_{01}^{TM}\right]_{ka=2.95}$. The input power is the power of the incident TM_{01} wave in the circular waveguide. $P_t + P_r = 1$.

With the value of x_{01} given in [5, page 2], (8.3) becomes

$$[\lambda_{01}^{TM}]_{ka=2.95} = \frac{2\pi a}{\sqrt{(2.95)^2 - (2.40482556)^2}} = 3.67738806a. \qquad (8.4)$$

The value $ka = 2.95$ was chosen because it is fairly central to the range of values of ka in Figs. 8.1 to 8.5 (see the next paragraph). The values of L_1, and L_2 do not matter because the loads Z_1 and Z_2 were chosen to be matched loads, that is,

$$Z_1 = Z_2 = Z_{10}^{TE}. \qquad (8.5)$$

The curves of Figs. 8.1 to 8.5 are plotted for the entire range of ka such that only the TE_{10} mode propagates in the rectangular waveguides and only the TM_{01} and TE_{11} modes propagate in the circular waveguide. Since only the TE_{10} mode propagates in the rectangular waveguides,

$$\pi < kb < \min\left(2\pi, \frac{\pi b}{c}\right) \qquad (8.6)$$

where "min" denotes the minimum of the values in parentheses. Since only the TM_{01} and TE_{11} modes propagate in the circular waveguide,

$$x_{01} < ka < x'_{21}. \tag{8.7}$$

Substituting (8.2) and (8.3) into (8.6), we obtain

$$2.85599332 < ka < 5.7119866. \tag{8.8}$$

Taking the values of x_{01} and x'_{21} given in [5, pages 2 and 32], (8.7) becomes

$$2.40482556 < ka < 3.05423693 \tag{8.9}$$

Combining (8.8) and (8.9), we have

$$2.85599332 < ka < 3.05423693. \tag{8.10}$$

In Figs. 8.1 to 8.5, P_t approaches zero as ka approaches 2.8559932. This is expected because the TE_{10} mode, which carries the transmitted power, ceases to propagate when ka becomes less than 2.8559932.

The numerical data of Figs. 8.1 to 8.5 were computed with

$$\text{BKM} = 15 \tag{8.11}$$

$$\text{XM} = 40. \tag{8.12}$$

The parameters BKM and XM are not written with italicized letters because they are input variables for our computer program. The parameter BKM is introduced in (A.1) and used in (5.2). The constraint (5.2) on the values of p and q determines the order of the moment matrix $[Y^1 + Y^2 + Y^3]$ which appears in [1, eq. (2.22)]. When BKM = 15, the order of the moment matrix is 32. The parameter XM is introduced in (B.4). The effect of XM is to truncate the doubly infinite sum $\sum_{r=0}^{\infty} \sum_{s=1}^{\infty}$ that appears in (3.1)-(3.4). The truncated sum is $\sum_{r=0}^{r_{\max}} \sum_{s=1}^{s_{\max}}$ where $s_{\max}$, which depends on r, is the largest value of s such that

$$\left. \begin{array}{ll} j_{0,s} \le \text{XM}, & r = 0 \\ j'_{4,s} \le \text{XM}, & r = 1, 2, \cdots \end{array} \right\}. \tag{8.13}$$

Assuming that $\text{XM} > j_{0,1}$, $r_{\max}$ is the largest value of r such that $j'_{r,1} \le \text{XM}$.

Some of the time-average incident power is transmitted into the rectangular waveguides. The rest of it is reflected in the circular waveguide. Therefore,

$$P_t + P_r = 1. \tag{8.14}$$

The plots of P_t and P_r shown in Figs. 8.1 to 8.5 do indeed satisfy (8.14). Because the values of P_t and P_r computed separately from (5.49) and (8.1) satisfy (8.12), we have some confidence in their accuracy.

The magnetic field due to the impressed source $\underline{J}^{\text{imp}}$ radiating in the circular waveguide with the apertures closed is $\underline{H}^{(3)}(\underline{J}^{\text{imp}}, \underline{0})$ given by [1, eq. (5.11)]. The electric current at $z = 0$ on the wall of the circular waveguide associated with this magnetic field is $\underline{J}$ given by

$$\underline{J} = -\underline{u}_z \frac{2e^{-j\beta_{01}^{TM} L_3}}{\sqrt{\pi}a} \cos(\beta_{01}^{TM} L_3). \tag{8.15}$$

The magnitude of $\underline{J}$ of (8.15) is maximum when $L_3 = 0.5\lambda_{01}^{TM}$ where λ_{01}^{TM} is the wavelength of the TM_{01} mode in the circular waveguide. If an aperture is put where the electric current would otherwise be maximum, the tangential electric field will be large in this aperture. A large aperture field gives a large transmitted power P_t. Thus, we expect P_t to be large at $ka = 2.95$ in Fig. 8.4 because L_3 is then equal to $0.5\lambda_{01}^{TM}$. Actually, P_t is even larger at $ka = 2.95$ in Figs. 8.2 and 8.3 where $L_3 = 0.4\lambda_{01}^{TM}$ and $0.45\lambda_{01}^{TM}$, respectively. When ka is held at 2.95, the curve of P_t versus L_3/λ_{01}^{TM} shown in Fig. 8.6 attains a maximum at a value of L_3/λ_{01}^{TM} somewhat less than 0.5. This phenomenon may be due to the finite extent of the aperture in the z-direction. Note that $P_t = 0$ at $L_3/\lambda_{01}^{TM} = 0.25$ in Fig. 8.6. This is expected because the magnitude of $\underline{J}$ of (8.15) vanishes when $L_3/\lambda_{01}^{TM} = 0.25$. In this case, the aperture has little effect because there is no flow of electric current to stop at $z = 0$.

The data for the plot of P_t versus L_3/λ_{01}^{TM} of Fig. 8.6 were computed with

$$\left.\begin{aligned} \frac{b}{a} &= 1.1 \\ \frac{c}{a} &= 0.5 \\ ka &= 2.95 \\ \text{BKM} &= 15 \\ \text{XM} &= 40 \\ Z_1 &= Z_2 = Z_{01}^{TE} \end{aligned}\right\}. \tag{8.16}$$

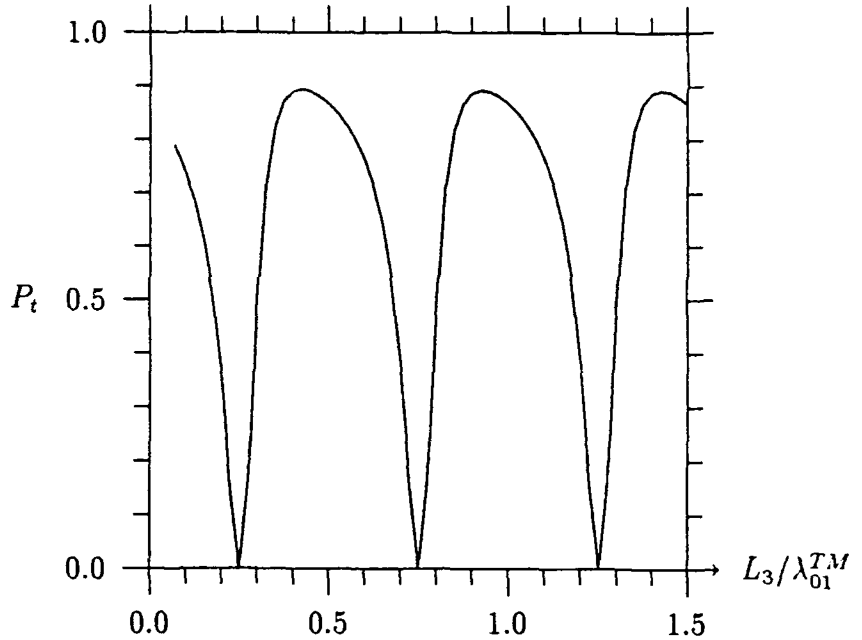


Fig. 8.6. Plot of the ratio P_t of the transmitted power to the input power versus L_3/λ_{01}^{TM} when $ka = 2.95$. The input power is the power of the incident TM_{01} wave in the circular waveguide.

The curve in Fig. 8.6 was terminated at $L_3/\lambda_{01}^{TM} = 1.5$. As L_3/λ_{01}^{TM} becomes larger and larger, P_t versus L_3/λ_{01}^{TM} becomes more and more periodic with period one. Adding one to the value of L_3/λ_{01}^{TM} does not change the reflection of the TM_{01} wave from the short at $z = L_3$. It only changes the reflections of the even and odd TE_{11} modes and all the nonpropagating modes. Now, with the parameters of (8.16), our solution for the electric field in the circular waveguide did not contain any TE_{11} modes. The computed values of the constants C_{11}^{TEe} and C_{11}^{TEo} in (6.75) were zero. When L_3/λ_{01}^{TM} is large, there is not much reflection of nonpropagating modes from the short at $z = L_3$ because any nonpropagating mode suffers attenuation on its journey from the aperture to the short at $z = L_3$. Furthermore, its reflection suffers the same amount of attenuation in going from the short at $z = L_3$ back to the aperture.

In Fig. 8.6, P_t could not be plotted for $L_3/\lambda_{01}^{TM} < 0.067983034$ because L_3 cannot be less than $c/2$. The approximate value 0.067983034 is obtained by writing

$$\frac{L_3}{\lambda_{01}^{TM}} = \frac{c}{2\lambda_{01}^{TM}} \tag{8.17}$$

and substituting (8.3) and (8.5) into the right-hand side of (8.17) to obtain

$$\frac{L_3}{\lambda_{01}^{TM}} = \frac{\sqrt{(2.95)^2 - (2.40482556)^2}}{8\pi} = 0.067983034. \tag{8.18}$$

Figures 8.7 to 8.13 show plots of $|E_\phi^{(A2)}(\phi,0)|/|\underline{E}_{01}^{TMe+}|_{\text{rms}}$ of (7.13) versus ϕ/ϕ_o and $|E_z^{(A2)}(0,z)|/|\underline{E}_{01}^{TMe+}|_{\text{rms}}$ versus z/c for ka = 2.855994, 2.86, 2.90, 2.95, 3.00, 3.05, and 3.054236 when $L_3/[\lambda_{01}^{TM}]_{ka=2.95} = 0.5$. As given by (7.13), $E_\phi^{(A2)}(\phi,0)/|\underline{E}_{01}^{TMe+}|_{\text{rms}}$ is the ϕ-component of the normalized electric field at $z = 0$ in the right-hand aperture A_2. As given by (7.14), $E_z^{(A2)}(0,z)/|\underline{E}_{01}^{TMe+}|_{\text{rms}}$ is the z-component of the normalized electric field at $\phi = 0$ in A_2. The values 2.855994 and 3.054236 were purposely chosen close to the lower and upper bounds in (8.10). The curves in Figs. 8.7 to 8.13 are not smooth because they were obtained by drawing straight lines between points spaced 0.025 apart in ϕ/ϕ_o and 0.05 apart in z/z_o. In Figs. 8.7 to 8.13, $|\underline{E}_z^{(A2)}(0,z)|$ is generally much larger than $|\underline{E}_\phi^{(A2)}(\phi,0)|$. This is expected because the aperture A_2 stops only the z-directed electric current $\underline{J}$ of (8.15). There is no ϕ-directed electric current to stop.

The data for the plots in Figs. 8.7 to 8.13 were computed with

$$\left.\begin{aligned} \frac{b}{a} &= 1.1 \\ \frac{c}{a} &= 0.5 \\ L_3 &= 0.5[\lambda_{01}^{TM}]_{ka=2.95} \\ \text{BKM} &= 33 \\ \text{XM} &= 100 \\ Z_1 &= Z_2 = Z_{01}^{TE} \end{aligned}\right\} \tag{8.19}$$

where $[\lambda_{01}^{TM}]_{ka=2.95}$ is given by (8.5). For the data in (8.19), the computed values of $V_{pq}^{\gamma TM}$ and $V_{pq}^{\gamma TE}$ satisfy

$$\left.\begin{aligned} V_{pq}^{1TM} &= V_{pq}^{2TM} \\ V_{pq}^{1TE} &= V_{pq}^{2TE} \end{aligned}\right\} \tag{8.20}$$

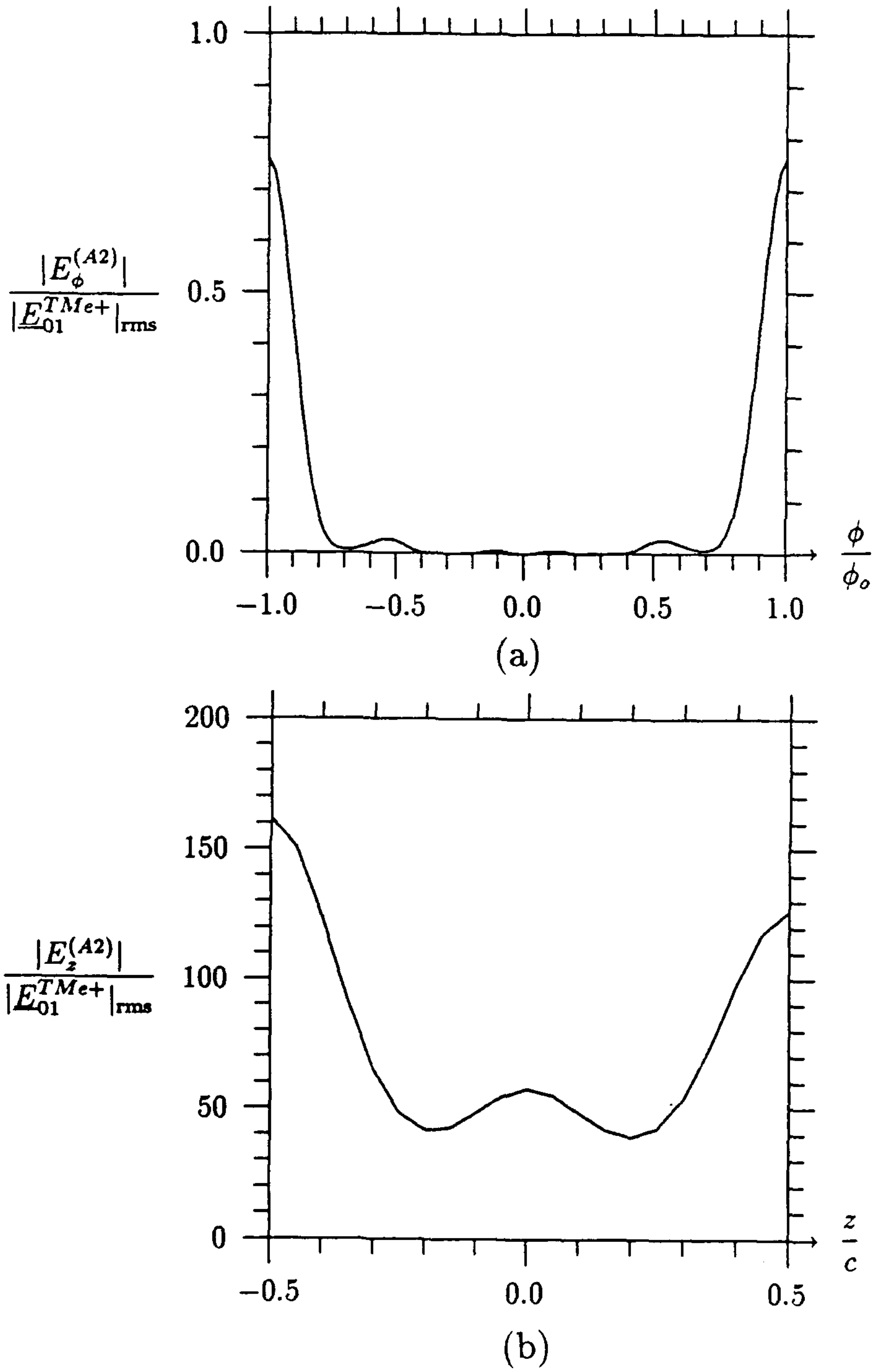


Fig. 8.7. The ratios (a) $|E_\phi^{(A2)}|/|\underline{E}_{01}^{TMe+}|_{\rm rms}$ and (b) $|E_z^{(A2)}|/|\underline{E}_{01}^{TMe+}|_{\rm rms}$ of the magnitudes of the ϕ- and z-directed electric fields in the aperture A_2 to the root mean square value of the electric field of the incident TM_{01} wave in the circular cylinder when $ka = 2.855994$ and $L_3 = 0.5\left[\lambda_{01}^{TM}\right]_{ka=2.95}$.

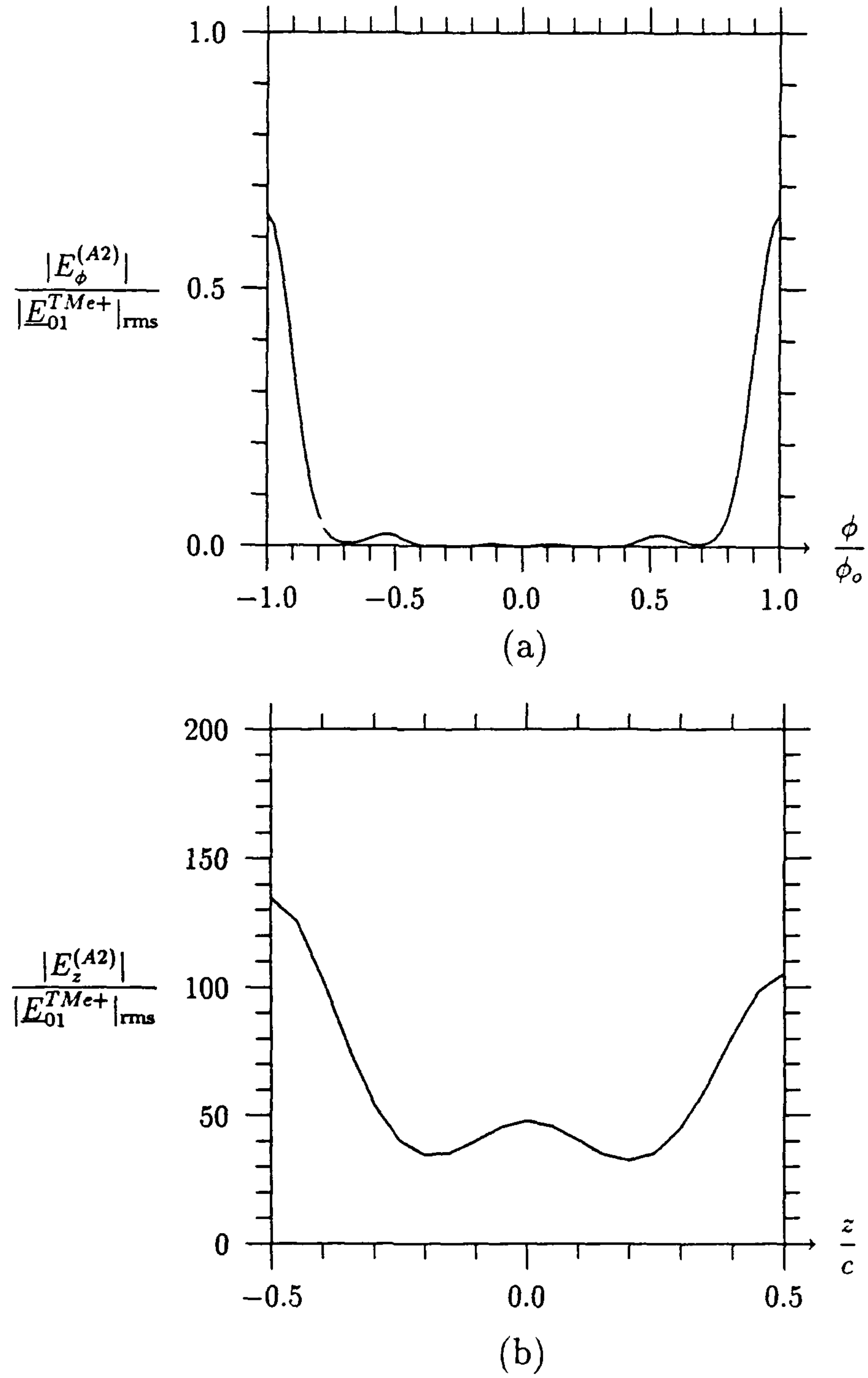


Fig. 8.8. The ratios (a) $|E_\phi^{(A2)}|/|\underline{E}_{01}^{TMe+}|_{\rm rms}$ and (b) $|E_z^{(A2)}|/|\underline{E}_{01}^{TMe+}|_{\rm rms}$ of the magnitudes of the ϕ- and z-directed electric fields in the aperture A_2 to the root mean square value of the electric field of the incident TM_{01} wave in the circular cylinder when $ka = 2.86$ and $L_3 = 0.5\left[\lambda_{01}^{TM}\right]_{ka=2.95}$.

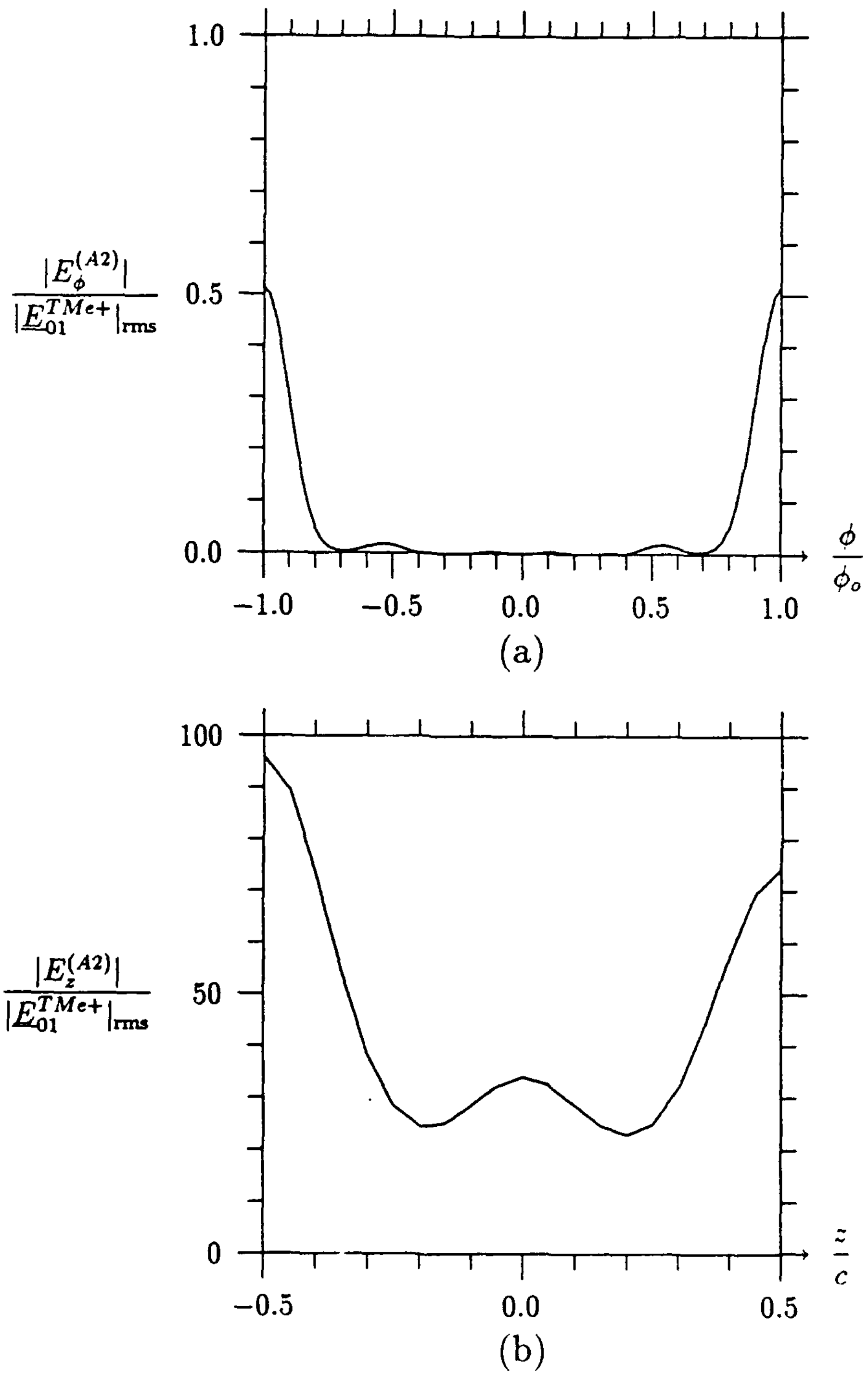
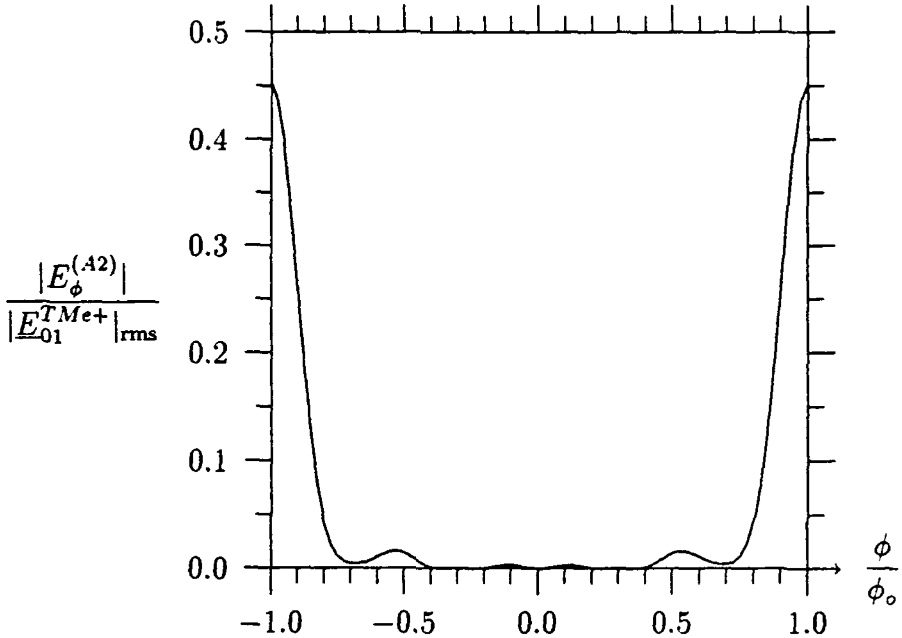
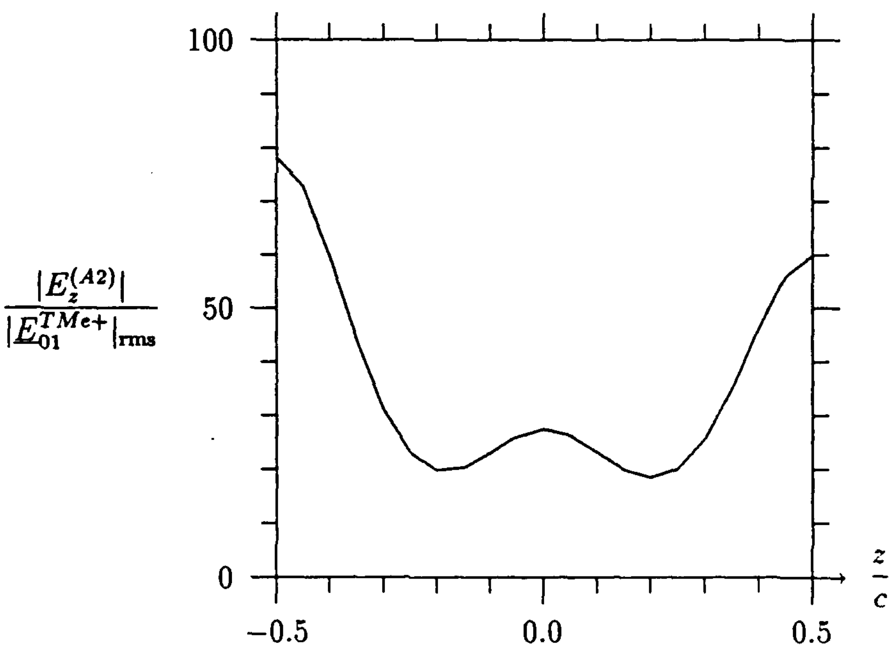


Fig. 8.9. The ratios (a) $|E_\phi^{(A2)}|/|E_{01}^{TMe+}|_{\mathrm{rms}}$ and (b) $|E_z^{(A2)}|/|E_{01}^{TMe+}|_{\mathrm{rms}}$ of the magnitudes of the ϕ- and z-directed electric fields in the aperture A_2 to the root mean square value of the electric field of the incident TM_{01} wave in the circular cylinder when $ka = 2.90$ and $L_3 = 0.5\left[\lambda_{01}^{TM}\right]_{ka=2.95}$.



(a)



(b)

Fig. 8.10. The ratios (a) $|E_\phi^{(A2)}|/|\underline{E}_{01}^{TMe+}|_{\rm rms}$ and (b) $|E_z^{(A2)}|/|\underline{E}_{01}^{TMe+}|_{\rm rms}$ of the magnitudes of the ϕ- and z-directed electric fields in the aperture A_2 to the root mean square value of the electric field of the incident TM_{01} wave in the circular cylinder when $ka = 2.95$ and $L_3 = 0.5\left[\lambda_{01}^{TM}\right]_{ka=2.95}$.

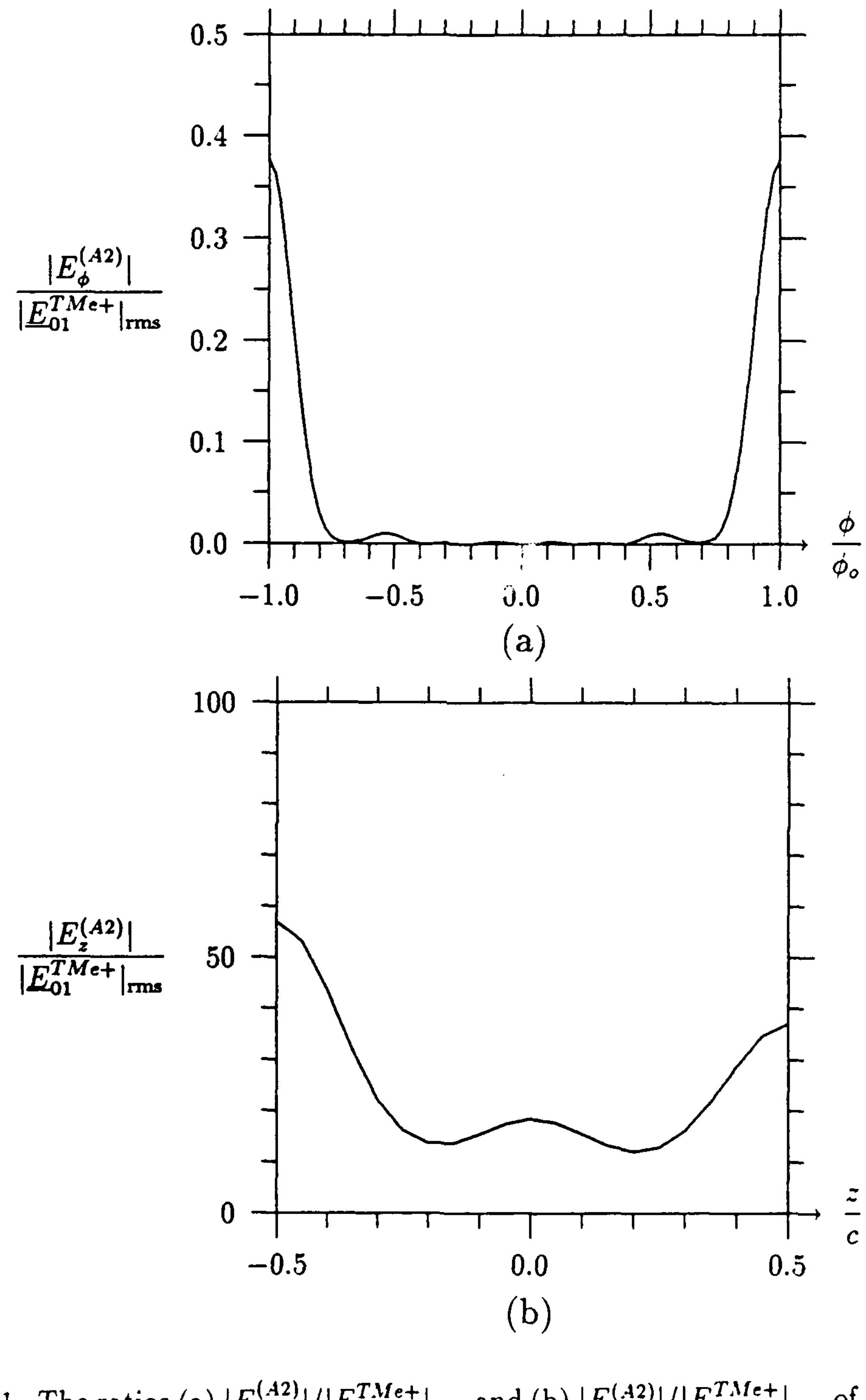


Fig. 8.11. The ratios (a) $|E_\phi^{(A2)}|/|E_{01}^{TMe+}|_{\text{rms}}$ and (b) $|E_z^{(A2)}|/|E_{01}^{TMe+}|_{\text{rms}}$ of the magnitudes of the ϕ- and z-directed electric fields in the aperture A_2 to the root mean square value of the electric field of the incident TM_{01} wave in the circular cylinder when $ka = 3.00$ and $L_3 = 0.5\left[\lambda_{01}^{TM}\right]_{ka=2.95}$.

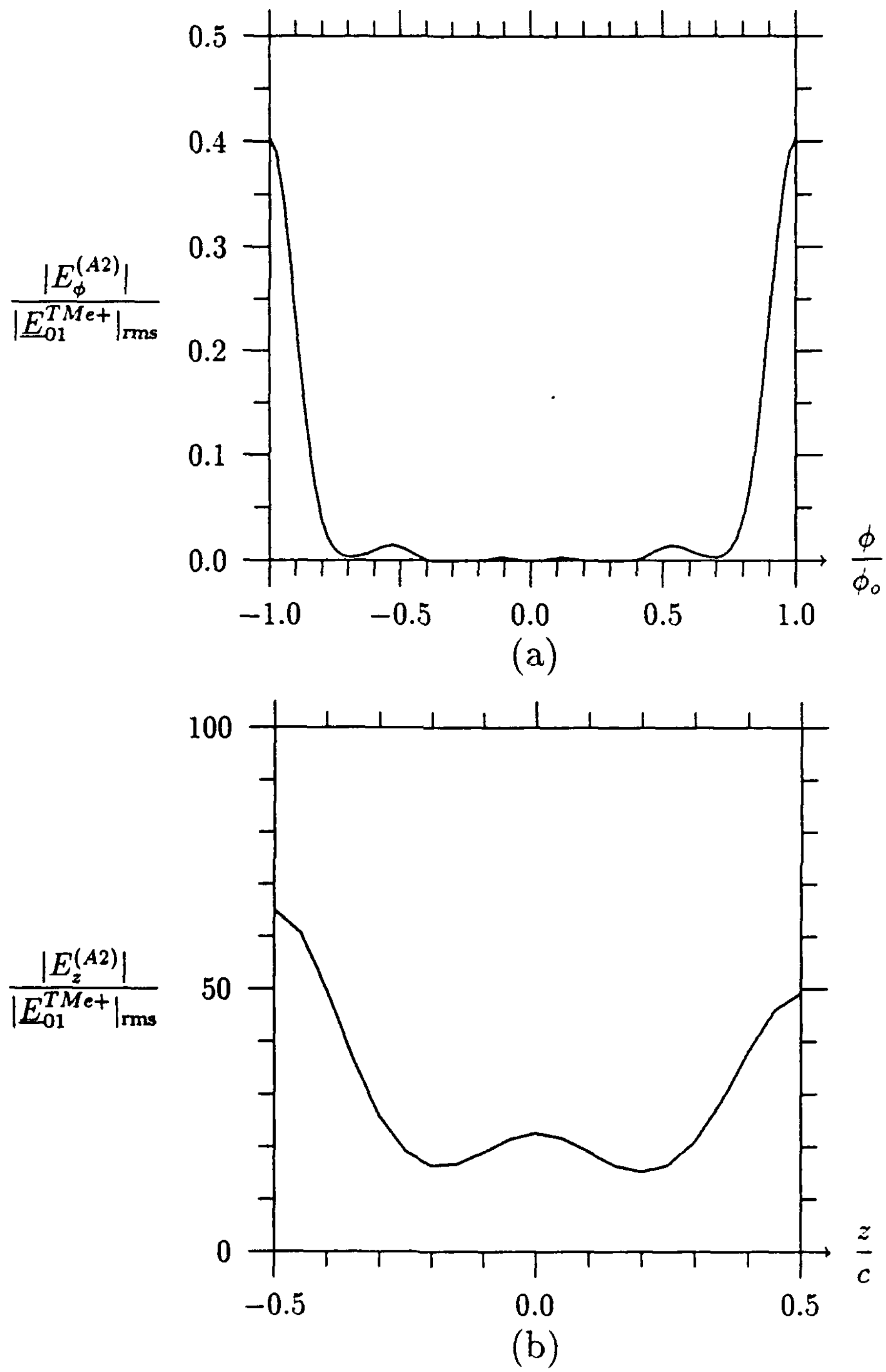


Fig. 8.12. The ratios (a) $|E_\phi^{(A2)}|/|\underline{E}_{01}^{TMe+}|_{rms}$ and (b) $|E_z^{(A2)}|/|\underline{E}_{01}^{TMe+}|_{rms}$ of the magnitudes of the ϕ- and z-directed electric fields in the aperture A_2 to the root mean square value of the electric field of the incident TM_{01} wave in the circular cylinder when $ka = 3.05$ and $L_3 = 0.5\left[\lambda_{01}^{TM}\right]_{ka=2.95}$.

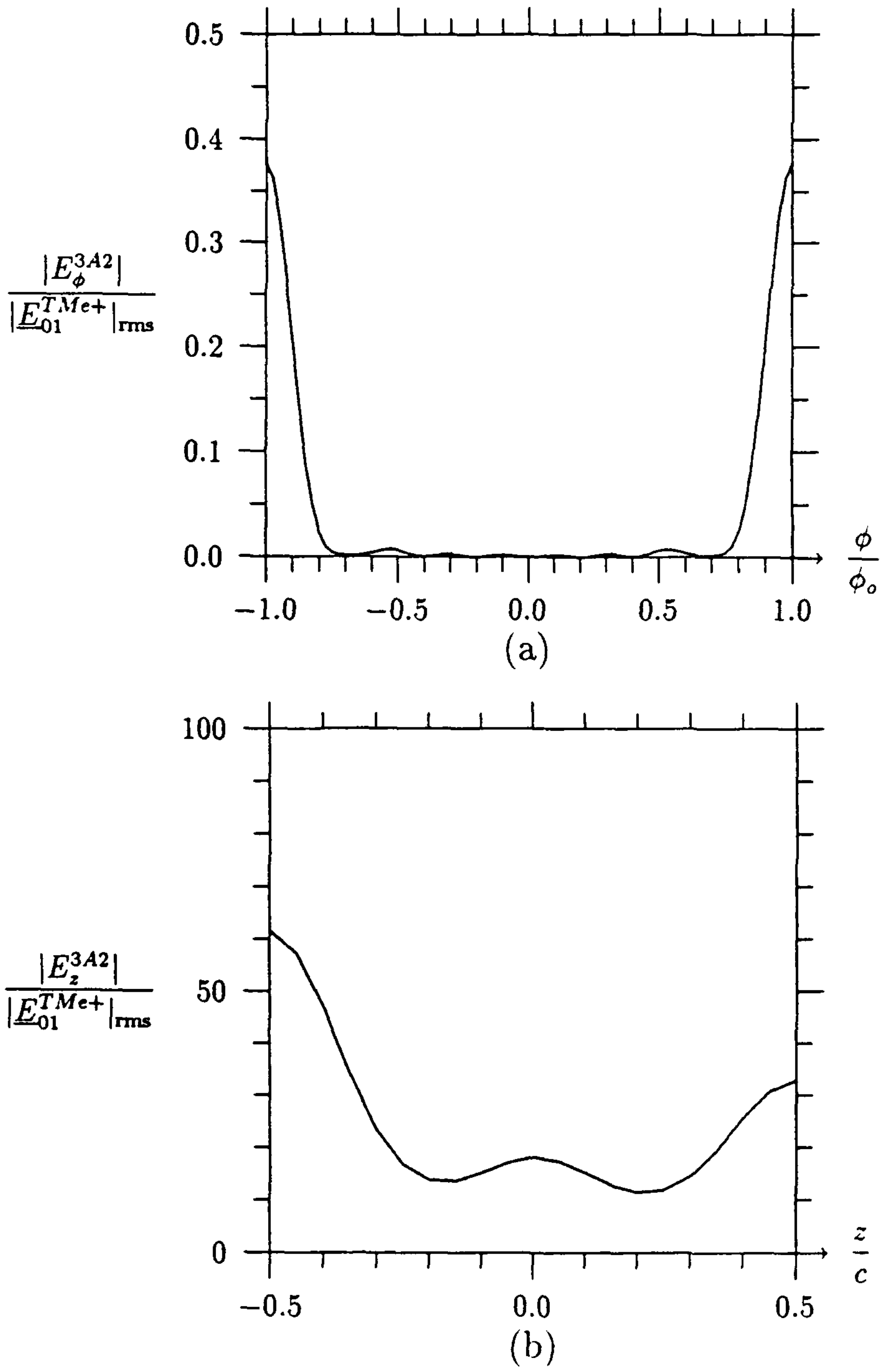


Fig. 8.13. The ratios (a) $|E_\phi^{(A2)}|/|\underline{E}_{01}^{TMe+}|_{\mathrm{rms}}$ and (b) $|E_z^{(A2)}|/|\underline{E}_{01}^{TMe+}|_{\mathrm{rms}}$ of the magnitudes of the ϕ- and z-directed electric fields in the aperture A_2 to the root mean square value of the electric field of the incident TM_{01} wave in the circular cylinder when $ka = 3.054236$ and $L_3 = 0.5\left[\lambda_{01}^{TM}\right]_{ka=2.95}$.

the angle $\pi - \phi$ in A_2. Furthermore, the z component of the electric field at the angle ϕ in A_1 is equal to the z component of the electric field at the angle $\pi - \phi$ in A_2.

Note that the values of BKM and XM in (8.19) are larger than those in (8.16). Larger values of BKM and XM were needed for accurate calculation of the tangential electric field in A_2 than for accurate calculation of P_t. Larger values were needed because the convergence of the electric field in A_2 with increasing BKM and XM was slower than the convergence of P_t. The electric field in A_2 converges slowly because its ϕ-component tends toward infinity as $1/(1-(\phi/\phi_o)^2)^\nu$ when ϕ/ϕ_o approaches ± 1 where $\nu = (1-2\phi_o/\pi)/(3-2\phi_o/\pi)$, and its z-component tends toward infinity as $1/(1-(2z/c)^2)^{1/3}$ when $2z/c$ approaches ± 1 [6, page 387].

Appendix A

Ordering the Expansion Functions

The expansion function $\underline{M}^{1\delta}_{mn}(\phi, z)$ for the equivalent magnetic current $\underline{M}^{(1)}$ is, according to [1, eq. (2.13)], defined in terms of $\underline{e}^{\delta}_{mn}$. Here, δ may be either TM or TE, $\underline{e}^{TM}_{mn}$ is the electric-type mode vector of the TM_{mn} rectangular waveguide mode, and $\underline{e}^{TE}_{mn}$ is the electric-type mode vector of the TE_{mn} rectangular waveguide mode. The expansion function $\underline{M}^{2\delta}_{mn}(\phi, z)$ is, according to [1, eq. (2.14)], also defined in terms of $\underline{e}^{\delta}_{mn}$.

The expansion functions are arranged in the following order:

1. $\left\{\underline{M}^{1TM}_{m1}, m = 1, 2, \cdots, \mathrm{MM}(2)\right\}$, $\left\{\underline{M}^{1TM}_{m2}, m = 1, 2, \cdots, \mathrm{MM}(3)\right\}$, $\left\{\underline{M}^{1TM}_{m3}, m = 1, 2, \cdots, \mathrm{MM}(4)\right\}, \cdots$
2. $\left\{\underline{M}^{1TE}_{m0}, m = 1, 2, \cdots, \mathrm{MM}(1)\right\}$, $\left\{\underline{M}^{1TE}_{m1}, m = 0, 1, \cdots, \mathrm{MM}(2)\right\}$, $\left\{\underline{M}^{1TE}_{m2}, m = 0, 1, \cdots, \mathrm{MM}(3)\right\}, \cdots$
3. $\left\{\underline{M}^{2TM}_{m1}, m = 1, 2, \cdots, \mathrm{MM}(2)\right\}$, $\left\{\underline{M}^{2TM}_{m2}, m = 1, 2, \cdots, \mathrm{MM}(3)\right\}$, $\left\{\underline{M}^{2TM}_{m3}, m = 1, 2, \cdots, \mathrm{MM}(4)\right\}, \cdots$
4. $\left\{\underline{M}^{2TE}_{m0}, m = 1, 2, \cdots, \mathrm{MM}(1)\right\}$, $\left\{\underline{M}^{2TE}_{m1}, m = 0, 1, \cdots, \mathrm{MM}(2)\right\}$, $\left\{\underline{M}^{2TE}_{m2}, m = 0, 1, \cdots, \mathrm{MM}(3)\right\}, \cdots$.

As listed above, items 3 and 4 are, respectively, items 1 and 2 with the superscript 1 replaced by 2.

The MM's in items 1, 2, 3, and 4 are variables that are calculated in our computer program. The value of MM $(n+1)$ is determined in the following manner. The sequences of expansion functions $\left\{\underline{M}_{mn}^{1TM}\right\}$, $\left\{\underline{M}_{mn}^{1TE}\right\}$, $\left\{\underline{M}_{mn}^{2TM}\right\}$, and $\left\{\underline{M}_{mn}^{2TE}\right\}$ are terminated by requiring that the nonnegative integers m and n be small enough so that

$$k_{mn}b \leq \mathrm{BKM} \tag{A.1}$$

where k_{mn} is the mode cutoff wavenumber given by (2.4), and BKM is a specified constant. The parameter BKM is not written with italicized letters because it is an input variable for our computer program. Substitution of (2.4) into (A.1) gives

$$\sqrt{(m\pi)^2 + \left(\frac{n\pi b}{c}\right)^2} \leq \mathrm{BKM}. \tag{A.2}$$

According to (A.2), MM $(n+1)$ is the largest integer such that

$$\sqrt{(\mathrm{MM}\,(n+1)\pi)^2 + \left(\frac{n\pi b}{c}\right)^2} \leq \mathrm{BKM}. \tag{A.3}$$

All values of n so large that no integer MM $(n+1)$ satisfies (A.3) are disallowed.

If we define $\underline{M}_i^{1TM}$ to be the i^{th} $\underline{M}^{1TM}$ in item 1 for $i = 1, 2, \cdots$, then there will be a pair of integers m and n such that $\underline{M}_i^{1TM} = \underline{M}_{mn}^{1TM}$. Thus, there will, for the $\underline{M}^{1TM}$'s, be a correspondence between each pair of integers m and n in use and each single integer i. Replacing m, n, and i by p, q, and j, respectively, there will, for the $\underline{M}^{1TM}$'s, be a correspondence between each pair of integers p and q in use and each single integer j. This is what is called in the sentence following (2.5) the correspondence between each pair of integers (p, q) used in [1, eq. (3.44)] and the subscript j in [1, eq. (3.44)].

If we define $\underline{M}_i^{1TE}$ to be the i^{th} $\underline{M}^{1TE}$ in item 2 for $i = 1, 2, \cdots$, then there will be a pair of integers m and n such that $\underline{M}_i^{1TE} = \underline{M}_{mn}^{1TE}$. Thus, there will, for the $\underline{M}^{1TE}$'s, be a correspondence between each pair of integers m and n in use and each single integer i. Replacing m, n, and i by p, q, and j, respectively, there will, for the $\underline{M}^{1TE}$'s, be a correspondence between each pair of integers p and q in use and each single integer j. This is what is called in the sentence following (2.5) the correspondence between each pair of integers (p, q) used in [1, eq. (3.47)] and the subscript j in [1, eq. (3.47)].

Appendix B

Roots of Bessel Functions and Their Derivatives

This appendix describes how numerical values of $\{j_{ns},\ s = 1, 2, \cdots, s_{\max};\ n = 0, 1, \cdots\}$ and $\{j'_{ns},\ s = 1, 2, \cdots, s_{\max};\ n = 0, 1, \cdots\}$ are obtained. Here, j_{ns} is the s^{th} root of J_n:

$$J_n(j_{ns}) = 0. \tag{B.1}$$

Furthermore, j'_{ns} is the s^{th} root of J'_n:

$$J'_n(j'_{ns}) = 0. \tag{B.2}$$

In (B.1), J_n is the Bessel function of the first kind of order n. In (B.2), J'_n is the derivative of J_n with respect to its argument. Now, the roots $\{j_{ns}\}$ and $\{j'_{ns}\}$ are ordered so that

$$\left.\begin{array}{l} 0 < j_{0,1} < j'_{0,1} < j_{0,2} < j'_{0,2} \cdots < j_{0,s_{\max}} < j'_{0,s_{\max}} \\ n < j'_{n,1} < j_{n,1} < j'_{n,2} < j_{n,2} \cdots < j'_{n,s_{\max}} < j_{n,s_{\max}},\quad n = 1, 2, \cdots \end{array}\right\} \tag{B.3}$$

Here, $s_{\max}$ depends on n. Given n, $s_{\max}$ is the largest value of s such that

$$\left.\begin{array}{ll} j_{0,s} \leq \mathrm{XM}, & n = 0 \\ j'_{n,s} \leq \mathrm{XM}, & n = 1, 2, \cdots \end{array}\right\} \tag{B.4}$$

where XM is not written with italicized letters because it is an input variable for our computer program. Assuming that $\mathrm{XM} > j_{0,1}$, all values of n so large that $j'_{n,1} > \mathrm{XM}$ are disallowed. In (B.4), XM is a constant that controls the number of roots that are calculated. If one wants to calculate more roots, one should choose a larger value of XM.

Our numerical values of $\{j_{ns},\ s = 1,2,\cdots,49;\ n = 0,1,\cdots,19\}$ and $\{j'_{ns},s = 1,2,\cdots,49;\ n = 0,1,\cdots,19\}$ are taken directly from [5, Tables 1 and 2]. For other values of n and s, we calculate j_{ns} and j'_{ns} by means of formulas given in [5]. In the body of the present report, the s^{th} roots of J_n and J'_n were called x_{ns} and x'_{ns} to coincide with the notation in [1]. Here in Appendix B, these roots are, more in harmony with the notation in [5], called j_{ns} and j'_{ns}. However, our notation in Appendix B is slightly different from that of [5]. Our j_{ns} is what Olver calls $j_{n,s}$. Our j'_{0s} is what Olver calls $j'_{0,s+1}$. Our j'_{ns} for $n \geq 1$ is what Olver calls $j'_{n,s}$. Our "inner fringe" calculated values of $\{j_{20s},\ s = 1,2,\cdots,50\}$, $\{j_{n50},\ n = 0,1,2,\cdots,19\}$, $\{j'_{20s},\ s = 1,2,\cdots,50\}$, and $\{j'_{n50},\ n = 0,1,2,\cdots,19\}$ agree well with the "outer fringe" tabulated values in [5, Tables 1 and 2]. There is no entry in [5, Table 2] which corresponds to our $j'_{0,50}$. However, according to [3, formula 9.1.28], $j'_{0,50} = j_{1,50}$, and there is an entry in [5, Table 1] which corresponds to our $j_{1,50}$. Although, as stated earlier in this paragraph, our notation does not normally place a comma between the subscripts of j and j', we had to use a comma in the previous two sentences in order to separate the "0" from the "50" and the "1" from the "50".

B.1 Evaluation of Roots of Bessel Functions of Large Order

For $n \geq 20$ and $s \geq 1$, we approximate j_{ns} by [5, eq. (9.01)]

$$j_{ns} = nz + \frac{p_1}{n} + \frac{p_2}{n^3} \tag{B.5}$$

where z, p_1, and p_2 are tabulated functions of $-\zeta$ where

$$-\zeta = -n^{-2/3}a_s. \tag{B.6}$$

In (B.6), a_s is the s^{th} negative root of the Airy function Ai:

$$\text{Ai}(a_s) = 0,\ \ s = 1,2,\cdots. \tag{B.7}$$

Here,

$$0 < -a_1 < -a_2 < -a_3 < \cdots. \tag{B.8}$$

The roots $\{a_s,\ s = 1, 2, \cdots, 50\}$ are tabulated [5, Table V, page 78]. For $s > 50$ [5, eq. (9.07)],

$$a_s = -\lambda^{2/3}\left(1 + \frac{5}{48\lambda^2}\right) \tag{B.9}$$

where [5, eq. (9.09)],

$$\lambda = \frac{3\pi}{8}(4s - 1)\,. \tag{B.10}$$

Actually, z, p_1, and p_2 are tabulated functions of $-\zeta$ only for $(0 \le -\zeta \le 7.5)$ [5, Table IV, pages 72 and 74]. If $-\zeta > 7.5$, then $z - 2/(3\xi^3)$ and p_1 are tabulated functions of ξ [5, Table IV, page 74] where

$$\xi = \frac{1}{\sqrt{-\zeta}}. \tag{B.11}$$

If $-\zeta > 7.5$, then $p_2 = 0$.

The modified interpolation formula of Everett [5, eq. (9.04)], [7, page 57],

$$f_p = (1 - p)f_0 + pf_1 + E_2\delta^2_{m0} + F_2\delta^2_{m1} + M_4\gamma^4_0 + N_4\gamma^4_1, \tag{B.12}$$

is used to obtain accurate values of z, p_1, and p_2. In (B.12), the f's are values of the function being interpolated, the δ^2_m's are modified second-order differences, and the γ^4's are modified fourth-order differences. In particular, f_p is the interpolated value of f at the actual value of the argument; f_0, δ^2_{m0}, and γ^4_0 are the tabulated values of f, δ^2_m, and γ^4 at the nearest smaller tabulated value of the argument; and f_1, δ^2_{m1}, and γ^4_1 are the tabulated values of f, δ^2_m, and γ^4 at the nearest larger tabulated value of the argument. The argument is either $-\zeta$ or ξ. Numerically, p is the ratio of the difference between the actual value and the nearest smaller tabulated value of the argument to the difference between the nearest larger and nearest smaller tabulated values of the argument. Thus, the actual value of the argument is a fraction p of the way from the nearest lower tabulated value to the nearest upper tabulated value. In (B.12), E_2, F_2, M_4, and N_4 are given by [7, pages 56 and 57]

$$E_2 = -\frac{p(p-1)(p-2)}{6} \tag{B.13}$$

$$F_2 = \frac{(p+1)p(p-1)}{6} \tag{B.14}$$

$$M_4 = 1000E_2\left\{\frac{(p+1)(p-3)}{20} + 0.184\right\} \tag{B.15}$$

$$N_4 = 1000F_2\left\{\frac{(p+2)(p-2)}{20} + 0.184\right\}. \tag{B.16}$$

B.2 Evaluation of Roots of Derivatives of Bessel Functions of Large Order

For $n \geq 20$ and $s \geq 1$, we approximate j'_{ns} by [5, eq. (9.02)]

$$j'_{ns} = nz + \frac{q_1}{n} + \frac{q_2}{n^3} + \frac{q_3}{n^5} \tag{B.17}$$

where z, q_1, q_2, and q_3 are tabulated functions of $-\zeta$ [5, Table IV, pages 72–75] where

$$-\zeta = -n^{-2/3}a'_s. \tag{B.18}$$

In (B.18), a'_s is the s^{th} negative root of Ai′, the derivative of the Airy function Ai:

$$\mathrm{Ai}'(a'_s) = 0, \quad s = 1, 2, \cdots. \tag{B.19}$$

Here,

$$0 < -a'_1 < -a'_2 < -a'_3 < \cdots. \tag{B.20}$$

The roots $\{a'_s,\ s = 1, 2, \cdots, 50\}$ are tabulated [5, Table V, page 78]. For $s > 50$ [5, eq. (9.08)],

$$a'_s = -\mu^{2/3}\left(1 - \frac{7}{48\mu^2}\right) \tag{B.21}$$

where [5, eq. (9.09)]

$$\mu = \frac{3\pi}{8}(4s - 3). \tag{B.22}$$

The tabulation of z was described in the two sentences following (B.10). Actually, q_1, q_2, and q_3 are tabulated functions of $-\zeta$ only for $0 \leq \zeta \leq 7.5$ [5, Table IV, pages 73 and 75]. If $-\zeta > 7.5$, then $q_2 = q_3 = 0$ and q_1 is a tabulated function of ξ [5, Table IV, page 75] where ξ is given by (B.11). The interpolation formula (B.12) is used to obtain accurate values of z, q_1, q_2, and q_3.

B.3 Evaluation of Large Roots of Bessel Functions

For $0 \leq n \leq 19$ and $s \geq 50$, we approximate j_{ns} and j'_{ns} by the truncated McMahon expansions [5, eqs. (1.10) and (1.12)]

$$j_{ns} = \beta - \sum_{r=1}^{4} \frac{A_{2r-1}}{(2r-1)!2^{3r}\beta^{2r-1}} \tag{B.23}$$

$$j'_{ns} = \beta' - \sum_{r=1}^{4} \frac{A'_{2r-1}}{(2r-1)!2^{3r}\beta'^{(2r-1)}} . \tag{B.24}$$

In (B.23),

$$\beta = (2n + 4s - 1)\frac{\pi}{4} \tag{B.25}$$

and [5, eq. (1.11)]

$$A_1 = \mu - 1 \tag{B.26}$$

$$A_3 = (\mu - 1)(7\mu - 31) \tag{B.27}$$

$$A_5 = 4(\mu - 1)(83\mu^2 - 982\mu + 3779) \tag{B.28}$$

$$A_7 = 6(\mu - 1)(6949\mu^3 - 153855\mu^2 + 1585743\mu - 6277237) \tag{B.29}$$

where

$$\mu = 4n^2. \tag{B.30}$$

In (B.24),

$$\beta' = (2n + 4s - 3)\frac{\pi}{4} \tag{B.31}$$

and [5, eq. (1.13)]

$$A'_1 = \mu + 3 \tag{B.32}$$

$$A'_3 = 7\mu^2 + 82\mu - 9 \tag{B.33}$$

$$A'_5 = 4(83\mu^3 + 2075\mu^2 - 3039\mu + 3537) \tag{B.34}$$

$$A'_7 = 6(6949\mu^4 + 296492\mu^3 - 1248002\mu^2 + 7414380\mu - 5853627) \tag{B.35}$$

where μ is given by (B.30). Expressions (B.23) and (B.24) expand to

$$j_{ns} = \beta - \frac{A_1}{8\beta} - \frac{A_3}{384\beta^3} - \frac{1}{15360\beta^5}\left(\frac{A_5}{4}\right) - \frac{1}{3440640\beta^7}\left(\frac{A_7}{6}\right) \tag{B.36}$$

$$j'_{ns} = \beta' - \frac{A'_1}{8\beta'} - \frac{A'_3}{384\beta'^3} - \frac{1}{15360\beta'^5}\left(\frac{A'_5}{4}\right) - \frac{1}{3440640\beta'^7}\left(\frac{A'_7}{6}\right) . \tag{B.37}$$

Appendix C

The Effect of Loads on the TM_{01} Source in the Circular Waveguide

In the body of this report, we find the electric fields in the waveguide mode converter due to the excitation of a unit-amplitude z-traveling TM_{01} wave. The source of this wave is taken to be the electric current source $\underline{J}^{\text{imp}}$ whose $-z$-traveling waves see a matched load. Specializing further, we take $\underline{J}^{\text{imp}}$ to be the transverse electric surface current density located at $z = -L_5$ that launches the z-traveling wave whose electromagnetic field is $(\underline{E}_{01}^{TMe+}, \underline{H}_{01}^{TMe+})$ in the region for which $z > -L_5$. Here,

$$\left.\begin{aligned} \underline{E}_{01}^{TMe+} &= Z_{01}^{TMeo}\underline{e}_{01}^{TMe}(\rho,\phi)e^{-j\beta_{01}^{TM}z} + \underline{u}_z\frac{(k_{01}^{TM})^2\psi_{01}^{TMe}(\rho,\phi)e^{-j\beta_{01}^{TM}z}}{j\omega\epsilon} \\ \underline{H}_{01}^{TMe+} &= \underline{h}_{01}^{TMe}(\rho,\phi)e^{-j\beta_{01}^{TM}z} \end{aligned}\right\}. \quad \text{(C.1)}$$

Equation (C.1) was obtained by substituting (6.63) into [1, eq. (B.1)]. The electric and magnetic fields defined by (C.1) are those of the z-traveling TM_{01} mode. We assume that $\underline{J}^{\text{imp}}$ also launches the $-z$-traveling wave whose electromagnetic field is $C(\underline{E}_{01}^{TMe-}, \underline{H}_{01}^{TMe-})$ in the region for which $z < -L_5$. Here, C is an unknown constant and

$$\left.\begin{aligned} \underline{E}_{01}^{TMe-} &= -Z_{01}^{TMeo}\underline{e}_{01}^{TMe}(\rho,\phi)e^{j\beta_{01}^{TM}z} + \underline{u}_z\frac{(k_{01}^{TM})^2\psi_{01}^{TMe}(\rho,\phi)e^{j\beta_{01}^{TM}z}}{j\omega\epsilon} \\ \underline{H}_{01}^{TMe-} &= \underline{h}_{01}^{TMe}(\rho,\phi)e^{j\beta_{01}^{TM}z} \end{aligned}\right\}. \quad \text{(C.2)}$$

Equation (C.2) was obtained by substituting (6.63) into [1, eq. (B.2)]. The electric and magnetic fields defined by (C.2) are those of the $-z$-traveling TM_{01} mode. Requiring the transverse electric field to be continuous at $z = -L_5$, we obtain

$$C = -e^{j2\beta_{01}^{TM} L_5}. \tag{C.3}$$

The TM_{01} waves radiated by $\underline{J}^{\mathrm{imp}}$ in the circular waveguide are shown in Fig. C.1 where "1" is the coefficient of the z-traveling TM_{01} mode in the region for which $-L_5 < z < -c/2$. "Γ" is the coefficient of the $-z$-traveling TM_{01} mode in the region for which $-L_5 < z < -c/2$. "$C + \Gamma$" is the coefficient of the $-z$-traveling TM_{01} mode in the region for which $-\infty < z < -L_5$. The TM_{01} waves shown in Fig. C.1 are those dealt with in the body of this report. Thus, from (6.75), we have

$$\Gamma = -C_{01}^{TMe} e^{-j2\beta_{01}^{TM} L_3}. \tag{C.4}$$

Here, Γ is a reflection coefficient because γ is the ratio of the coefficient of the $-z$-traveling mode field (C.2) to the coefficient of the z-traveling mode field (C.1) in the region for which $-L_5 < z < -c/2$. Since the magnetic fields of the mode fields are $\underline{h}_{01}^{TMe}(\rho,\phi)e^{\mp j\beta_{01}^{TM} z}$ rather than $\pm\underline{h}^{TMe}(\rho,\phi)e^{\mp j\beta_{01}^{TM} z}$, γ is a reflection coefficient for the current rather than for the voltage.

The loads mentioned in the title of this appendix are taken to be the TM_{01} loads Z_{L4} at $z = -L_4$ and Z_{L6} at $z = -L_6$. See Fig. C.2. A TM_{01} load is a load that acts on the voltage and current of only the TM_{01} waves. When the loads Z_{L4} and Z_{L6} are in place, the TM_{01} electric and magnetic fields $\underline{E}$ and $\underline{H}$ in the circular waveguide are, as indicated in Fig. C.2, assumed to be given by

$$\left.\begin{aligned} \underline{E} &= C_4^+ \underline{E}_{01}^{TMe+} + C_4^- \underline{E}_{01}^{TMe-} \\ \underline{H} &= C_4^+ \underline{H}_{01}^{TMe+} + C_4^- \underline{H}_{01}^{TMe-} \end{aligned}\right\} \quad -L_4 < z- < L_5 \tag{C.5}$$

$$\left.\begin{aligned} \underline{E} &= C_5^+ \underline{E}_{01}^{TMe+} + C_5^- \underline{E}_{01}^{TMe-} \\ \underline{H} &= C_5^+ \underline{H}_{01}^{TMe+} + C_5^- \underline{H}_{01}^{TMe-} \end{aligned}\right\} \quad -L_5 < z < -L_6 \tag{C.6}$$

$$\left.\begin{aligned} \underline{E} &= C_6^+ \underline{E}_{01}^{TMe+} + C_6^- \underline{E}_{01}^{TMe-} \\ \underline{H} &= C_6^+ \underline{H}_{01}^{TMe+} + C_6^- \underline{H}_{01}^{TMe-} \end{aligned}\right\} \quad -L_6 < z < -\frac{c}{2} \tag{C.7}$$

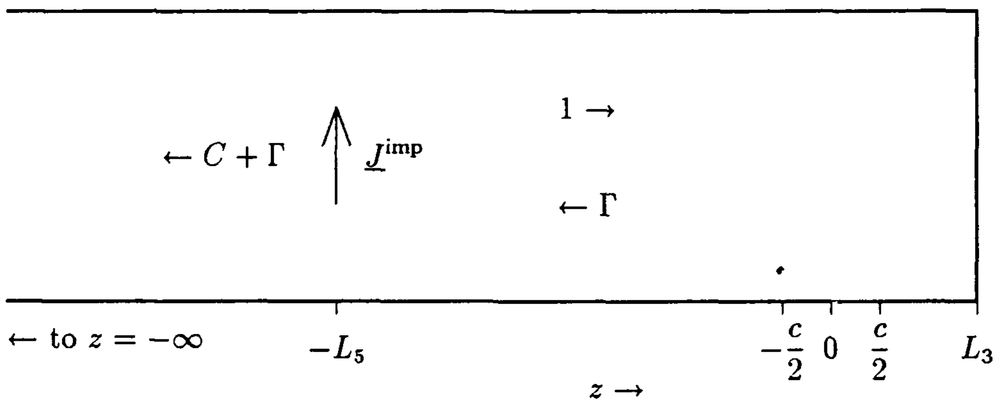


Fig. C.1. The TM_{01} waves radiated by $\underline{J}^{\text{imp}}$ in the circular waveguide. The situation in Fig. C.1 is the same as that in the body of this report. There are no additional loads.

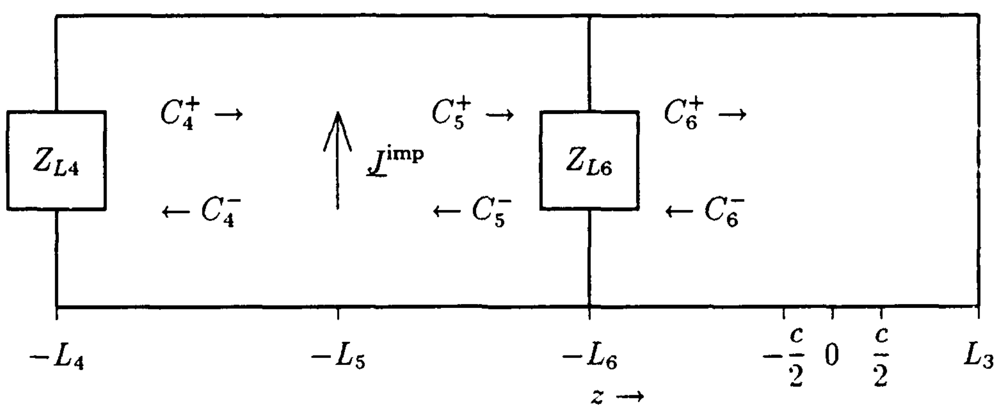


Fig. C.2. The TM_{01} waves radiated by $\underline{J}^{\text{imp}}$ when the additional loads Z_{L4} and Z_{L6} are present in the circular waveguide.

where the C's are unknown constants.

The effect of the loads Z_{L4} and Z_{L6} is to change the electric field incident on the aperture-perforated section of circular waveguide from $\underline{E}_{01}^{TMe+}$ in Fig. C.1 to $C_6^+ \underline{E}_{01}^{TMe+}$ in Fig. C.2. Viewing this incident electric field as the excitation of the aperture-perforated section of circular waveguide, we deduce that the fields in this section of waveguide in Fig. C.2 are those in Fig. C.1 multiplied by C_6^+. Therefore,

$$C_6^- = \Gamma C_6^+. \tag{C.8}$$

Equation (C.8) is one simultaneous equation in the variables C_4^+, C_4^-, C_5^+, C_5^-, C_6^+, and C_6^-. In the following five paragraphs, we obtain five more simultaneous equations in these six variables.

Because the waves launched by $\underline{J}^{\text{imp}}$ in Fig. C.2 are the same as those in Fig. C.1, we have

$$C_5^+ = C_4^+ + 1 \tag{C.9}$$

$$C_4^- = C_5^- + C. \tag{C.10}$$

Recall that the loads Z_{L4} and Z_{L6} act on the voltages and currents of the TM_{01} waves. These voltages and currents are called the TM_{01} voltages and currents. Seeking to define the TM_{01} voltages and currents, we substitute (C.1) and (C.2) into (C.5)-(C.7) and take only the transverse part of $\underline{E}$, which is called $\underline{E}_t$:

$$\left.\begin{aligned} \underline{E}_t &= \left(C_4^+ e^{-j\beta_{01}^{TM} z} - C_4^- e^{j\beta_{01}^{TM} z}\right) Z_{01}^{TMeo} \underline{e}_{01}^{TMe}(\rho,\phi) \\ \underline{H} &= \left(C_4^+ e^{-j\beta_{01}^{TM} z} + C_4^- e^{j\beta_{01}^{TM} z}\right) \underline{h}_{01}^{TMe}(\rho,\phi) \end{aligned}\right\} - L_4 < z < -L_5 \tag{C.11}$$

$$\left.\begin{aligned} \underline{E}_t &= \left(C_5^+ e^{-j\beta_{01}^{TM} z} - C_5^- e^{j\beta_{01}^{TM} z}\right) Z_{01}^{TMeo} \underline{e}_{01}^{TMe}(\rho,\phi) \\ \underline{H} &= \left(C_5^+ e^{-j\beta_{01}^{TM} z} + C_5^- e^{j\beta_{01}^{TM} z}\right) \underline{h}_{01}^{TMe}(\rho,\phi) \end{aligned}\right\} - L_5 < z < -L_6 \tag{C.12}$$

$$\left.\begin{aligned} \underline{E}_t &= \left(C_6^+ e^{-j\beta_{01}^{TM} z} - C_6^- e^{j\beta_{01}^{TM} z}\right) Z_{01}^{TMeo} \underline{e}_{01}^{TMe}(\rho,\phi) \\ \underline{H} &= \left(C_6^+ e^{-j\beta_{01}^{TM} z} + C_6^- e^{j\beta_{01}^{TM} z}\right) \underline{h}_{01}^{TMe}(\rho,\phi) \end{aligned}\right\} - L_6 < z < -\frac{c}{2}. \tag{C.13}$$

The TM_{01} voltages are defined to be the coefficients of $\underline{e}_{01}^{TMe}(\rho,\phi)$ in (C.11)-(C.13). The TM_{01} currents are defined to be the coefficients of $\underline{h}_{01}^{TMe}(\rho,\phi)$ in

(C.11)-(C.13). Viewing Fig. C.2 as a circuit, the TM_{01} voltage is the voltage of the upper line with respect to the lower one, and the TM_{01} current is the z-directed electric current on the upper line.

The presence of Z_{L4} at $z = -L_4$ requires that

$$\left[\frac{V}{I}\right]_{z=-L_4^+} = -Z_{L4} \tag{C.14}$$

where V is the TM_{01} voltage and I is the TM_{01} current. The subscript "$z = -L_4^+$" denotes the limit as z approaches $-L_4$ from above. "From above" means through values which are greater than $-L_4$. Extracting V and I from (C.11) and substituting them into (C.14), we obtain

$$\frac{C_4^+ e^{jl_4} - C_4^- e^{-jl_4}}{C_4^+ e^{jl_4} + C_4^- e^{-jl_4}} = -\frac{Z_{L4}}{Z_{01}^{TMeo}} \tag{C.15}$$

where

$$l_4 = \beta_{01}^{TM} L_4. \tag{C.16}$$

Solving (C.15) for C_4^+ in terms of C_4^-, we arrive at

$$C_4^+ = \Gamma_4 C_4^- \tag{C.17}$$

where

$$\Gamma_4 = \frac{Z_{01}^{TMeo} - Z_{L4}}{Z_{01}^{TMeo} + Z_{L4}} e^{-j2l_4}. \tag{C.18}$$

Since there is no series load at $z = -L_6$,

$$[V]_{z=-L_6^-} = [V]_{z=-L_6^+} \tag{C.19}$$

where the subscript "$z = -L_6^-$" denotes the the limit as z approaches $-L_6$ from below and "$z = -L_6^+$" denotes the limit as z approaches $-L_6$ from above. Extracting the V's from (C.12) and (C.13) and substituting them into (C.19), we obtain

$$C_5^+ e^{jl_6} - C_5^- e^{-jl_6} = C_6^+ e^{jl_6} - C_6^- e^{-jl_6} \tag{C.20}$$

where

$$l_6 = \beta_{01}^{TM} L_6. \tag{C.21}$$

At $z = -L_6$, the TM_{01} current V/Z_{L6} flows from the upper terminal of Z_{L6} to its lower terminal so that

$$[I]_{z=-L_6^-} = \frac{1}{Z_{L6}}[V]_{z=-L_6^-} + [I]_{z=-L_6^+}. \tag{C.22}$$

Extracting V and I from (C.12) and (C.13) and substituting them into (C.22), we obtain

$$C_5^+ e^{jl_6} + C_5^- e^{-jl_6} = C_6^+ e^{jl_6} + C_6^- e^{-jl_6} + \frac{C_5^+ e^{jl_6} - C_5^- e^{-jl_6}}{Z} \tag{C.23}$$

where

$$Z = \frac{Z_{L6}}{Z_{01}^{TMeo}}. \tag{C.24}$$

Equation (C.23) becomes

$$C_5^+(1-Z)e^{jl_6} - C_5^-(1+Z)e^{-jl_6} + C_6^+ Z e^{jl_6} + C_6^- Z e^{-jl_6} = 0. \tag{C.25}$$

Equations (C.17), (C.10), (C.9), (C.20), (C.25), and (C.8), ordered as cited, are written in matrix form as

$$\begin{bmatrix} 1 & -\Gamma_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -e^{jl_6} & e^{-jl_6} & e^{jl_6} & -e^{-jl_6} \\ 0 & 0 & (1-Z)e^{jl_6} & -(1+Z)e^{-jl_6} & Ze^{jl_6} & Ze^{-jl_6} \\ 0 & 0 & 0 & 0 & -\Gamma & 1 \end{bmatrix} \begin{bmatrix} C_4^+ \\ C_4^- \\ C_5^+ \\ C_5^- \\ C_6^+ \\ C_6^- \end{bmatrix} = \begin{bmatrix} 0 \\ C \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{C.26}$$

We proceed to solve the matrix equation (C.26) for C_4^+, C_4^-, C_5^+, C_5^-, C_6^+, and C_6^-.

Adding the first row of (C.26) and the product of Γ_4 with the second row to the third row, we obtain, in view of (C.3),

$$\begin{bmatrix} 1 & -\Gamma_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -\Gamma_4 & 0 & 0 \\ 0 & 0 & -e^{jl_6} & e^{-jl_6} & e^{jl_6} & -e^{-jl_6} \\ 0 & 0 & (1-Z)e^{jl_6} & -(1+Z)e^{-jl_6} & Ze^{jl_6} & Ze^{-jl_6} \\ 0 & 0 & 0 & 0 & -\Gamma & 1 \end{bmatrix} \begin{bmatrix} C_4^+ \\ C_4^- \\ C_5^+ \\ C_5^- \\ C_6^+ \\ C_6^- \end{bmatrix} = \begin{bmatrix} 0 \\ C \\ D \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{C.27}$$

where

$$D = 1 - \Gamma_4 e^{j2\beta_{01}^{TM} L_5}. \tag{C.28}$$

The last four rows of (C.27) are

$$\begin{bmatrix} 1 & - & 0 & 0 \\ -e^{jl_6} & e^{-jl_6} & e^{jl_6} & -e^{-jl_6} \\ (1-Z)e^{jl_6} & -(1+Z)e^{-jl_6} & Ze^{jl_6} & Ze^{-jl_6} \\ 0 & 0 & -\Gamma & 1 \end{bmatrix} \begin{bmatrix} C_5^+ \\ C_5^- \\ C_6^+ \\ C_6^- \end{bmatrix} = \begin{bmatrix} D \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{C.29}$$

Adding the product of e^{jl_6} with the first row to the second row of (C.29) and adding the product of $-(1-Z)e^{jl_6}$ with the first row to the third row, we obtain

$$\begin{bmatrix} 1 & -\Gamma_4 & 0 & 0 \\ 0 & e^{-jl_6} - \Gamma_4 e^{jl_6} & e^{jl_6} & -e^{-jl_6} \\ 0 & -(1+Z)e^{-jl_6} + (1-Z)\Gamma_4 e^{jl_6} & Ze^{jl_6} & Ze^{-jl_6} \\ 0 & 0 & -\Gamma & 1 \end{bmatrix} \begin{bmatrix} C_5^+ \\ C_5^- \\ C_6^+ \\ C_6^- \end{bmatrix} = \begin{bmatrix} D \\ De^{jl_6} \\ -(1-Z)De^{jl_6} \\ 0 \end{bmatrix}. \tag{C.30}$$

Discarding the first row of (C.30) and multiplying the second and third rows by e^{-jl_6}, we obtain

$$\begin{bmatrix} e^{-j2l_6} - \Gamma_4 & 1 & -e^{-j2l_6} \\ -(1+Z)e^{-j2l_6} + (1-Z)\Gamma_4 & Z & Ze^{-j2l_6} \\ 0 & -\Gamma & 1 \end{bmatrix} \begin{bmatrix} C_5^- \\ C_6^+ \\ C_6^- \end{bmatrix} = \begin{bmatrix} D \\ -(1-Z)D \\ 0 \end{bmatrix}. \quad \text{(C.31)}$$

Adding the product of the first row with $\{(1+Z)e^{-j2l_6} - (1-Z)\Gamma_4\}/\{e^{-j2l_6} - \Gamma_4\}$ to the second row of (C.31), we obtain

$$\begin{bmatrix} e^{-j2l_6} - \Gamma_4 & 1 & -e^{-j2l_6} \\ 0 & \dfrac{(1+2Z)e^{-j2l_6} - \Gamma_4}{e^{-j2l_6} - \Gamma_4} & \dfrac{-e^{-j4l_6} - (2Z-1)\Gamma_4 e^{-j2l_6}}{e^{-j2l_6} - \Gamma_4} \\ 0 & -\Gamma & 1 \end{bmatrix} \begin{bmatrix} C_5^- \\ C_6^+ \\ C_6^- \end{bmatrix} = \begin{bmatrix} D \\ \dfrac{2ZDe^{-j2l_6}}{e^{-j2l_6} - \Gamma_4} \\ 0 \end{bmatrix}. \quad \text{(C.32)}$$

Discarding the first row of (C.32) and multiplying the second row by $e^{-j2l_6} - \Gamma_4$, we obtain

$$\begin{bmatrix} (1+2Z)e^{-j2l_6} - \Gamma_4 & -e^{-j4l_6} - (2Z-1)\Gamma_4 e^{-j2l_6} \\ -\Gamma & 1 \end{bmatrix} \begin{bmatrix} C_6^+ \\ C_6^- \end{bmatrix} = \begin{bmatrix} 2ZDe^{-j2l_6} \\ 0 \end{bmatrix}. \quad \text{(C.33)}$$

Adding the product of the first row of (C.33) with $\Gamma/\{(1+2Z)e^{-j2l_6} - \Gamma_4\}$ to the second row, we obtain

$$\left\{1 - \frac{\Gamma(e^{-j4l_6} + (2Z-1)\Gamma_4 e^{-j2l_6})}{(1+2Z)e^{-j2l_6} - \Gamma_4}\right\} C_6^- = \frac{2ZD\Gamma e^{-j2l_6}}{(1+2Z)e^{-j2l_6} - \Gamma_4}. \quad \text{(C.34)}$$

Solving (C.34) for C_6^-, we obtain

$$C_6^- = \frac{2ZD\Gamma}{\Delta} \tag{C.35}$$

where

$$\Delta = 1 + 2Z - (2Z-1)\Gamma\Gamma_4 - \Gamma_4 e^{j2l_6} - \Gamma e^{-j2l_6}. \tag{C.36}$$

Substitution of (C.35) into the second row of (C.33) gives

$$C_6^+ = \frac{2ZD}{\Delta}. \tag{C.37}$$

Substituting (C.35) and (C.36) into the first row of (C.32) and solving for C_5^-, we arrive at

$$C_5^- = \frac{\{(2Z-1)\Gamma + e^{j2l_6}\}D}{\Delta}. \tag{C.38}$$

Substitution of (C.38) into the first row of (C.30) gives

$$C_5^+ = \frac{\{2Z + 1 - \Gamma e^{-j2l_6}\}D}{\Delta}. \tag{C.39}$$

Next, C_4^- is given by (C.10). Finally, C_4^+ is given by (C.17). It can be verified that the C's given by (C.35), (C.37), (C.38), (C.39), (C.10), and (C.17) do indeed satisfy (C.26).

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